A decomposition of the class of semiconvex games

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Abstract

The paper studies the class $\mathbf{SC^N}$ of cooperative games with player set N which have the semiconvexity property. The class of semiconvex games is decomposed into an algebraic sum of convex cones of games for which generating sets are available. The union of these generating sets forms a generating set for $\mathbf{SC^N}$. Special attention is paid to the games in the latter set. In particular, the so called airport savings games w_x, $x \in IR^N$, defined by $w_x(S) = \sum_{j \in S} x_j - \max_{j \in S} x_j$ for each non-empty subset S of the player set N, are emphasized.

The semiconvex games form a subset of the class of the semibalanced games. The generating sets of the latter class and $\mathbf{SC^N}$ wil be used here to study the relationships between these two classes. Also the class of games with a non-empty core and the quasi-balanced games will be treated in this context.

1 Submodular cost and supermodular savings functions

As an application of game theoretic analysis to the cost allocation problem, Littlechild and Owen (1973) studied the problem of setting airport landing charges for different types of aircraft. Their game theoretic approach to the

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airport cost allocation problem is based on an appropriately defined function, the so-called airport cost function $c : 2^N \rightarrow I\!R$. Here N is the set of planes which are to land at the airport and the cost function c is determined by the landing of the largest plane in a given subset of planes. It is known that the airport cost function c satisfies the submodularity condition

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T) \quad \text{for all } S, T \subseteq N.$$

As usual, an arbitrary cost function $c : 2^N \rightarrow I\!R$ with respect to a finite set N induces a savings function $v : 2^N \rightarrow I\!R$ by means of $v(\emptyset) = 0$ and

$$v(S) = \sum_{j \in S} c(\{j\}) - c(S) \quad \text{for all } S \subseteq N, \;\; S \neq \emptyset.$$

Here, the expression $v(S)$ represents the cost savings that would result in the cost model if the participants in subset S cooperate instead of acting alone. Whenever the cost function c satisfies the submodularity condition, then the induced savings function v satisfies the supermodularity condition

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all } S, T \subseteq N. \tag{1}$$

The savings function arising from the airport cost function can be put in a more general framework. Consider the situation in which n persons are at a taxi rank. Each person's home is situated on the route that the taxi-driver drives to take the person with the most expensive ride home. For each person i, $1 \leq i \leq n$, let x_i denote the cost to get home on his own by taxicab. With the cost vector $x = (x_1, x_2, \ldots, x_n) \in I\!R^n$ we associate the savings function $w_x : 2^N \rightarrow I\!R$ by means of $N = \{1, 2, \ldots, n\}$, $w_x(\emptyset) = 0$ and

$$w_x(S) = \sum_{j \in S} x_j - \max_{j \in S} x_j \quad \text{for all } S \subseteq N, S \neq \emptyset. \tag{2}$$

Here the expression $w_x(S)$ represents the cost savings that would result from sharing the taxicab by the persons in the subset S. It can easily be verified that the savings function w_x of (2) satisfies the supermodularity condition (1).

In the game theoretic context, the term convexity is preferred to the term supermodularity. This paper focuses on a related condition, the semiconvexity, which is a weaker condition than the convexity.

The paper has the following contents. In Section 2 the notion of semiconvexity for a cooperative game is described. Section 3 is devoted to a decomposition of the class of semiconvex games into an algebraic sum of three convex cones of games and, subsequently, a generating set for the class of semiconvex games is presented. Especially, this particular decomposition is strongly based on the smallest convex cone of games containing the savings games of (2). In Section 4 we will discuss both the Shapley value and the τ-value concept on the class of semiconvex games. The concluding section provides a figure which elucidates the relationships between the class of semiconvex games and three related classes of games.

2 The class of semiconvex games

First let us briefly go into the game theoretic setting. We consider a cooperative game in characteristic function form, or simply a *game*, with finite player set N to be a real-valued function v on the set 2^N of subsets of N with $v(\emptyset) = 0$. A subset S of N is called a *coalition* and $v(S)$ is called its *value* in the game v. The value $v(S)$ is interpreted as the gain or savings of the coalition S in the case the members of S decide to cooperate. The class of all games with player set N is denoted by $\mathbf{G^N}$. Note that $\mathbf{G^N}$ is the $(2^{|N|} - 1)$-dimensional Euclidean vector space indexed by the non-empty coalitions. The player set N is always supposed to consist of at least four players, i.e. $|N| \geq 4$.

An element $x = (x_i)_{i \in N}$ of $\mathbb{R}^N$ is called an *allocation*. With the allocation $x \in \mathbb{R}^N$ we associate the game $x : 2^N \to \mathbb{R}$ by means of $x(\emptyset) = 0$ and $x(S) = \sum_{j \in S} x_j$ for all $S \subseteq N$, $S \neq \emptyset$. Games which are associated in this way with an allocation are called *additive*. The class of all additive games with player set N is denoted by $\mathbf{A^N}$. Notice that $\mathbf{A^N}$ is an $|N|$-dimensional linear subspace of $\mathbf{G^N}$.

We say the allocation $x \in \mathbb{R}^N$ majorizes the game $v \in \mathbf{G^N}$ (notation: $x \geq v$) if the additive game associated with the allocation x majorizes the game v, i.e. $x(S) \geq v(S)$ for all $S \subseteq N$. Throughout the paper we pay special attention to the marginal contribution allocation of a game. Given an arbitrary game $v \in \mathbf{G^N}$ the corresponding *marginal contribution allocation* $b^v \in \mathbb{R}^N$ is defined by $b_i^v = v(N) - v(N \backslash \{i\})$ for all $i \in N$.

One of the main topics of research in cooperative game theory is how to allocate the value $v(N)$ in a game $v \in \mathbf{G^N}$ among the players. Since the introduction of the notion of a cooperative game, many solution concepts for these games have been proposed to solve the relevant allocation problem. Generally speaking, the concepts provide satisfactory and stable solutions only on a specific subclass of the game space. Perhaps the best-known solution concept is the so-called core: an allocation $x \in I\!R^N$ is said to be a *core allocation* of a game $v \in \mathbf{G^N}$ if x distributes the value $v(N)$ among the players in such a way that x majorizes v. In other words, the *core* $C(v)$ of a game $v \in \mathbf{G^N}$ consists of all allocations $x \in I\!R^N$ satisfying $x(N) = v(N)$ and $x \geq v$. Obviously, there are games without any core allocations.

The solution part of cooperative game theory is mainly based on the traditional assumption that the grand coalition N will be formed. Note that the marginal contribution allocation b^v of a game v is derived from the marginal contributions of each player with respect to the formation of the grand coalition. In Derks (1989) it is argued that the formation of the grand coalition in a game can only be expected whenever the game possesses the semibalancedness property. Here a game v is called *semibalanced* if the marginal contribution allocation b^v majorizes the game v, i.e. $b^v \geq v$. Semibalancedness is a necessary condition for the nonemptiness of the core since for each game v and core allocation $x \in C(v)$ we have

$$x_i = x(N) - x(N\backslash\{i\}) \leq v(N) - v(N\backslash\{i\}) = b^v_i \quad \text{for all } i \in N \qquad (3)$$

and, therefore, $v(S) \leq x(S) \leq b^v(S)$ for each coalition $S \subseteq N$, implying the semibalancedness of v.

It may happen that the semibalancedness inequalities $b^v(S) \geq v(S)$, $S \subseteq N$, even hold whenever an arbitrary marginal contribution b^v_i in the sum $b^v(S)$ is replaced by the individual value $v(\{i\})$ of the player i involved. Remark that we interpret the marginal contribution of any player as a large payoff to the player and the individual value as a small payoff. Due to these reasonings, the class of *semiconvex games* is defined by

$$\begin{aligned}\mathbf{SC^N} = \{v \in \mathbf{G^N} :\ & b^v \geq v \text{ and } b^v(S\backslash\{i\}) + v(\{i\}) \geq v(S) \\ & \qquad\qquad \text{for all } i \in N, \text{ all } S \subseteq N \text{ with } i \in S\}.\end{aligned}$$

Semiconvex games were introduced in Driessen and Tijs (1985) as an adjunct to the study of the τ-value concept. There it was also established that $\mathbf{SC^N}$ is a $(2^{|N|}-1)$-dimensional cone in $\mathbf{G^N}$ which includes the class of convex games (i.e. games satisfying the supermodularity condition (1)). Consequently, the savings game w_x of (2) associated with an arbitrary allocation $x \in I\!R^N$ is semiconvex. The structure of the class of semiconvex games is studied in the next section.

3 A decomposition of the class $\mathbf{SC^N}$

Our main goal is to decompose the class of semiconvex games into an algebraic sum of convex cones of games for which generating sets are available. The union of these generating sets forms a generating set for $\mathbf{SC^N}$.

As mentioned before, the savings games w_x of (2) arising from allocations $x \in I\!R^N$ are semiconvex. The class of these savings games, however, is not a convex cone as is illustrated by the following example. Consider the player set $N = \{1, 2, 3\}$ (the example can easily be extended to arbitrarily large N) and the savings games w_x and w_y associated with the allocations $x = (0, 1, 2)$ and $y = (2, 1, 0)$. In fact, the games w_x, w_y and the sum game $w = w_x + w_y$ are as follows.

coalition S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
value $w_x(S)$	0	0	0	0	0	0	1	1
value $w_y(S)$	0	0	0	0	1	0	0	1
value $w(S)$	0	0	0	0	1	0	1	2

Suppose w equals w_z for a certain allocation $z \in I\!R^3$. Then we have for any two-person coalition S $w(S) = w_z(S) = \sum_{j \in S} z_j - \max_{j \in S} z_j = \min_{j \in S} z_j$. From this and $w(\{1, 3\}) = 0$, we first derive that $z_1 = 0$ or $z_3 = 0$ and subsequently, $w(\{1, 2\}) \leq 0$ or $w(\{2, 3\}) \leq 0$; a contradiction. We assert that the sum game w is not a savings game.

Now the idea is to embed the class of these savings games in a convex cone generated by a finite number of appropriately defined savings games. For any coalition S let the allocation $1\!\!1^S \in I\!R^N$ be given by $1\!\!1^S_i = 1$ for all $i \in S$ and $1\!\!1^S_i = 0$ for all $i \in N \backslash S$. For the sake of notation, the savings game associated with the allocation $1\!\!1^S$ is shortly denoted by $w_S : 2^N \to I\!R$. Thus,

w_S is given by

$$w_S(T) = \begin{cases} |S \cap T| - 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We note that for coalitions S with $|S| \geq 2$ the marginal contribution allocation b^{w_S} of the savings game w_S equals the original allocation $1\!\!1^S$ and in case $|S| \leq 1$ then the savings game w_S equals the zero game and, thus, $b^{w_S} = 0$.

The next theorem states that each savings game associated with a non-negative allocation is a non-negative linear combination of the savings games w_S, $S \subseteq N$, i.e. the class of non-negative savings games is embedded in the convex cone $\mathbf{W^N}$ generated by the savings games w_S, $S \subseteq N$.

Theorem 1 *Let* w_x *be the savings game of (2), with* $x \in I\!R_+^N$. *Then* $w_x \in \mathbf{W^N} = \text{Cone}(\{w_S : S \subseteq N\})$.

Proof : Without loss of generality, the players may be ordered so that $N = \{1, 2, \ldots, n\}$ and $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$. For any $j \in N$, define the set $S_j = \{j, j+1, \ldots, n\}$ of players with index j or a larger index. It is evident that the vector equality $x = \sum_{j=1}^n (x_j - x_{j-1}) 1\!\!1^{S_j}$ holds. Now we assert that

$$w_x = \sum_{j=1}^n (x_j - x_{j-1}) w_{s_j}. \tag{5}$$

To prove (5) let $T \subseteq N$, $T \neq \emptyset$. Further, let $m \in T$ be the player in T with the largest index. Then we get $S_j \cap T = \emptyset$ iff $m < j \leq n$. Now it follows that

$$\begin{aligned} \sum_{j=1}^n (x_j - x_{j-1}) w_{s_j}(T) &= \sum_{j=1}^m (x_j - x_{j-1}) \left[|S_j \cap T| - 1\right] \\ &= \sum_{j=1}^m (x_j - x_{j-1}) \left[\sum_{k=j}^m 1\!\!1_k^T\right] - \sum_{j=1}^m (x_j - x_{j-1}) \\ &= \sum_{j=1}^m \sum_{k=j}^m (x_j - x_{j-1}) 1\!\!1_k^T - (x_m - x_0) \\ &= \sum_{k=1}^m \sum_{j=1}^k (x_j - x_{j-1}) 1\!\!1_k^T - x_m \\ &= \sum_{k=1}^m x_k 1\!\!1_k^T - x_m \end{aligned}$$

$$= \sum_{k \in T} x_k - \max_{k \in T} x_k \quad = \quad w_x(T).$$

We conclude that (5) holds. This completes the proof of the theorem. □

From the fact that the class of semiconvex games is a convex cone containing the savings games w_S, $S \subseteq N$, we conclude that the cone $\mathbf{W^N}$ is included in $\mathbf{SC^N}$. Without going into details, we note that any savings game w_x of (2) possesses a large core because of the supermodularity (convexity) property for the game w_x (cf. Sharkey 1982).

Next, we present a subclass of semiconvex games with a unique core allocation.

Lemma 2 *Let the class* $\mathbf{A_*^N}$ *of games be defined by*

$$\mathbf{A_*^N} = \{v \in \mathbf{G^N} :\ (v(\{i\}))_{i \in N} \textit{ is a (unique) core allocation of } v\}.$$

Then $\mathbf{A_*^N} \subset \mathbf{SC^N}$.

Proof: According to (3) each core allocation is majorized by the marginal contribution vector. In particular, $v \in \mathbf{A_*^N}$ yields that $b_i^v \geq v(\{i\})$ for all $i \in N$. From this we deduce that for all $i \in N$ and all $S \subsetneq N$ with $i \in S$ $b^v(S \backslash \{i\}) + v(\{i\}) \geq \sum_{j \in S} v(\{j\}) \geq v(S)$. Thus, $v \in \mathbf{SC^N}$. So, $v \in \mathbf{A_*^N}$ implies $v \in \mathbf{SC^N}$. □

Regarding semiconvex games with an empty core, we introduce the game $\hat{w} \in \mathbf{G^N}$ defined by

$$\hat{w}(S) = \begin{cases} -1 & \text{if } S \subseteq N,\ |S| \geq |N| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\hat{w}(N) = -1$ and $\hat{w}(\{i\}) = 0$ for all $i \in N$ because of $|N| \geq 3$. Hence, $\hat{w}$ has an empty core. But nevertheless, the game $\hat{w}$ is semiconvex for which the marginal contribution allocation $b^{\hat{w}}$ equals the zero allocation.

So far we have established that the class of semiconvex games contains the cones $\mathbf{W^N}$, $\mathbf{A_*^N}$ and the game $\hat{w}$. Due to the fact that $\mathbf{SC^N}$ itself is a convex cone, we conclude that the inclusion $\mathbf{SC^N} \supseteq \mathbf{A_*^N} + \mathbf{W^N} + \text{Cone}(\{\hat{w}\})$ holds. According to the main theorem, the inverse inclusion is also valid.

Theorem 3 $\mathbf{SC^N} = \mathbf{A_*^N} + \mathbf{W^N} + \text{ Cone}(\{\hat{w}\})$.

Proof: Let $v \in \mathbf{SC^N}$. We distinguish two cases.
Case one. Suppose that the game v is zero-normalized, i.e. $v(\{i\}) = 0$ for all $i \in N$. Then the semiconvexity of v yields that $b_i^v \geq v(\{i\}) = 0$ for all $i \in N$. From $b^v \geq 0$ and theorem 1 we derive that the savings game w_{b^v} belongs to the cone $\mathbf{W^N}$. For the real number $\alpha = b^v(N) - v(N) - \max_{j \in N} b_j^v$ we consider the game $w = v - w_{b^v} - \alpha\hat{w}$. Once again, the semiconvexity of v yields that $\alpha \geq 0$. Thus, $\alpha\hat{w} \in \text{ Cone}(\{\hat{w}\})$. Concerning the game w, we assert that $w(S) \leq 0$ for all $S \subseteq N$. Indeed,

$$\begin{array}{rcl}
w(\{i\}) & = & v(\{i\}) \quad = \quad 0 \quad \text{for all } i \in N, \\
w(N) & = & v(N) - b^v(N) + \max\limits_{j \in N} b_j^v + \alpha \quad = \quad 0, \\
w(N \backslash \{i\}) & = & v(N \backslash \{i\}) - b^v(N \backslash \{i\}) + \max\limits_{j \in N \backslash \{i\}} b_j^v + \alpha \\
 & = & \max\limits_{j \in N \backslash \{i\}} b_j^v - \max\limits_{j \in N} b_j^v \\
 & \leq & 0 \quad \text{for all } i \in N, \quad \text{and} \\
w(S) & = & v(S) - b^v(S) + \max\limits_{j \in S} b_j^v \\
 & \leq & 0 \quad \text{for all } S \subset N \text{ with } |S| < |N| - 1,
\end{array}$$

where the last inequality results from the semiconvexity and the zero-normalizedness of v. Hence, $w(S) \leq 0$ for all $S \subseteq N$. Together with $w(N) = 0$ and $w(\{i\}) = 0$ for all $i \in N$, this implies that the zero allocation is a core allocation of w and, therefore, $w \in \mathbf{A_*^N}$. We conclude that $v = w + w_{b^v} + \alpha\hat{w} \in \mathbf{A_*^N} + \mathbf{W^N} + \text{ Cone}(\{\hat{w}\})$.
Case two. Evidently, the additive game $x \in \mathbf{G^N}$ associated with the allocation $x = (v(\{i\}))_{i \in N}$ also belongs to the class $\mathbf{A_*^N}$. Furthermore, it is straightforward to verify that the semiconvexity of v implies the semiconvexity of the game $v - x$. By applying case one to the semiconvex zero-normalized game $v - x$, we obtain that $v - x \in \mathbf{A_*^N} + \mathbf{W^N} + \text{ Cone}(\{\hat{w}\})$. From this and $x \in \mathbf{A_*^N}$, we conclude that $v = v - x + x \in \mathbf{A_*^N} + \mathbf{W^N} + \text{ Cone}(\{\hat{w}\})$ for all $v \in \mathbf{SC^N}$. $\square$

The above mentioned theorem provides a decomposition of the class of semiconvex games into an algebraic sum of three convex cones of games. With the aid of the theorem we are able to present a generating set for $\mathbf{SC^N}$.

By its definition, the cone $\mathbf{W^N}$ is generated by the set $\{w_S : S \subseteq N, |S| \geq 2\}$. A generating set for the cone $\mathbf{A^N_*}$ can be found by noting that $\mathbf{A^N_*}$ is the algebraic sum of the class $\mathbf{A^N}$ of additive games and the class

$$\{v \in \mathbf{G^N} : \ v(N) = 0,\, v(\{i\}) = 0 \ \text{ for all } i \in N \text{ and } \\ v(S) \leq 0 \ \text{ for all other } S \subset N\}.$$

The latter class is simply generated by the set $\{-1_S : \ S \subset N,\ 2 \leq |S| \leq |N| - 1\}$ of games, where the *unity* game $1_S \in \mathbf{G^N}$ is given by

$$1_S(T) = 1 \ \text{ if } T = S \quad \text{and} \quad 1_S(T) = 0 \ \text{ for all } T \neq S.$$

The class $\mathbf{A^N}$ is generated by the set $\{u_i,\ -u_i\colon\ i \in N\}$ of games, where the *dictator* game $u_i \in \mathbf{G^N}$ is given by

$$u_i(T) = 1 \ \text{ if } T \ni i \quad \text{and} \quad u_i(T) = 0 \ \text{ otherwise.}$$

Corollary 4 *A generating set for the class* $\mathbf{SC^N}$ *is formed by the games*

$$\begin{array}{ll} u_i, -u_i, & i \in N, \\ -1_S, & S \subset N \textit{ with } 2 \leq |S| \leq |N| - 1, \\ w_S, & S \subseteq N \textit{ with } 2 \leq |S| \leq |N|, \quad \textit{and the game } \ \hat{w}. \end{array}$$

4 Solution concepts on the class $\mathbf{SC^N}$

We will discuss both the Shapley value and the τ-value concept on the class of semiconvex games. Because all the players in the game $\hat{w}$ are substitutes of each other, the Shapley value allocation $\phi(\hat{w})$ distributes the value $\hat{w}(N)$ equally among the players and, therefore, $\phi(\hat{w})$ equals $-|N|^{-1}\mathbb{1}^N \in \mathbb{R}^N$.

There is not much to be said about the Shapley value allocation of a game $v \in \mathbf{A^N_*}$ since the exact value of any multiperson coalition is rather arbitrary.

The determination and interpretation of the Shapley value allocation of a savings game w_x of (2) is similar to that of an airport cost game as treated in Littlechild and Owen (1973). In fact, the Shapley value concept on the class of these savings games is directly related to the following simple cost allocation rule. Let $x \in \mathbb{R}^N$ and suppose that the players are ordered so that $N = \{1, 2, \ldots, n\}$ and $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n$. Divide the smallest cost x_1 equally among all the players. Divide the incremental cost $x_2 - x_1$

(above the smallest cost x_1) equally among all the players except for the smallest player 1. Continue this procedure thus until the incremental cost $x_n - x_{n-1}$ is allocated to the largest player n. The Shapley value allocation is determined by the difference of the cost allocation x itself and the cost allocation resulting from the above mentioned simple rule.

Due to the linearity property of the Shapley value concept, the Shapley value allocation of an arbitrary semiconvex game can be determined as a non-negative linear combination of the Shapley value allocations of the games in the generating set for the class $\mathbf{SC^N}$ (see corollary 4).

Tijs (1981) introduced the τ-value concept on the class of quasi-balanced games. A game $v \in \mathbf{G^N}$ is said to be *quasi-balanced* if it satisfies the following two conditions:

1. the marginal contribution allocation b^v majorizes the *maximal remainder allocation* a^v which is defined by

$$a_i^v = \max_{S \ni i} [v(S) - b^v(S \backslash \{i\})] \text{ for all } i \in N.$$

2. there exists an allocation $y \in I\!R^N$ lying on the straight line segment with end points a^v and b^v, satisfying the additional efficiency requirement $y(N) = v(N)$.

The class of all quasi-balanced games with player set N is denoted by $\mathbf{Q^N}$. If the game v is quasi-balanced, the efficient allocation y on the line segment $[a^v, b^v]$ is unique and it is called the *τ-value allocation* $\tau(v)$ of the game v. For a detailed study of the τ-value concept on the class $\mathbf{Q^N}$ we refer to Driessen (1988, Chapter III).

Obviously, each semiconvex game v satisfies the condition $b^v \geq a^v$ because the semiconvexity of v yields that $b_i^v \geq v(\{i\}) = a_i^v$ for all $i \in N$. The second quasi-balancedness condition, however, may not be fulfilled by a semiconvex game as is illustrated by the game $\hat{w}$. In fact, $\hat{w}$ is not quasi-balanced since $b^{\hat{w}} = a^{\hat{w}} = 0$, whereas $\hat{w}(N) \neq 0$.

It turns out that the two cones $\mathbf{A_*^N}$ and $\mathbf{W^N}$ are included in $\mathbf{Q^N}$ and the τ-value concept on these two cones can be described as follows. The τ-value of a game $v \in \mathbf{A_*^N}$ coincides with the maximal remainder allocation a^v because $a^v = (v(\{i\}))_{i \in N}$ satisfies the efficiency requirement. Further, the τ-value allocation of a game $v \in \mathbf{A_*^N}$ is the unique core allocation and hence, it also coincides with the nucleolus allocation. Due to the fact that a savings game w_x of (2) is zero-normalized, its τ-value is proportional to the marginal contribution allocation b^{w_x}. If the maximum among the costs x_i, $i \in N$, is attained for more than one player, then the marginal contribution allocation b^{w_x} equals the cost allocation x and therefore, the τ-value of the savings game w_x is proportional to the allocation x itself. If there is a unique player j, with x_j being the largest cost , then the τ-value of the savings game w_x is proportional to the allocation x in which the largest cost x_j is replaced by the second largest cost.

As stated before, the two cones $\mathbf{A}_*^{\mathbf{N}}$ and $\mathbf{W}^{\mathbf{N}}$ are included in both $\mathbf{SC}^{\mathbf{N}}$ and $\mathbf{Q}^{\mathbf{N}}$. The intersection of $\mathbf{SC}^{\mathbf{N}}$ and $\mathbf{Q}^{\mathbf{N}}$, however, unequals the sum $\mathbf{A}_*^{\mathbf{N}} + \mathbf{W}^{\mathbf{N}}$ as is illustrated by the four-person game v given by $v(S) = 0, 1, -1, 0$ for $|S| = 1, 2, 3, 4$ respectively. We obtain that $b^v = 1\!\!1^N$, $a^v = 0$ and thus, the game v is both semiconvex and quasi-balanced. Nevertheless, the game v cannot be an element of the sum $\mathbf{A}_*^{\mathbf{N}} + \mathbf{W}^{\mathbf{N}}$ because its core is empty.

5 The class $\mathbf{SC}^{\mathbf{N}}$ compared with related classes of games

Figure 1 elucidates the relationships between the considered classes of games in more detail. Each point in the figure represents a cone in the game space $\mathbf{G}^{\mathbf{N}}$. This representation is as follows.

A: $\text{Cone}(\{-1_S\text{: } S \subset N,\ 2 \leq |S| \leq |N| - 2\}) + \mathbf{A}^{\mathbf{N}}$
B: $\text{Cone}(\{-1_{\{i\}}\text{: } i \in N\})$
C: $\text{Cone}(\{-1_{N \setminus \{i\}}\text{: } i \in N\})$
D: $\text{Cone}(\{\hat{w}\})$
E: $\text{Cone}(\{w_S\text{: } S \subseteq N,\ |S| \geq 2\}) = \mathbf{W}^{\mathbf{N}}$
F: $\text{Cone}(\{v_S\text{: } S \subseteq N,\ |S| \geq 2\})$.

For any coalition S with $|S| \geq 2$, let the game $v_S \in \mathbf{G}^{\mathbf{N}}$ be given by $v_S = w_S + (|S| - 1)\hat{w}$. Consequently, the games v_S, $S \subseteq N$ with $|S| \geq 2$, are included in the cone generated by the game $\hat{w}$ and the savings games w_S, $S \subseteq N$ with $|S| \geq 2$. Therefore, point **F** in figure 1 is located on the straight line segment with end points **D** and **E**. Generally speaking, any convex hull of points in figure 1 corresponds to the convex hull of the cones which are represented by the points concerned.

Note that point A represents the algebraic sum of the class of additive games and the cone generated by the games -1_S, $S \subseteq N$ with $2 \leq |S| \leq |N| - 2$. These games have been gathered in one cone since they fulfil each of the properties semibalancedness, quasi-balancedness, semiconvexity and moreover, each game in this cone possesses a nonempty core. The class $\mathbf{B}^{\mathbf{N}}$ of games with a nonempty core is actually the convex hull of the cones represented by the points **A**, **B** and **C** (see Derks 1989). Clearly, the triangle $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{C}\})$ contains point **E** because each savings game w_S, $S \subseteq N$, possesses a nonempty core.

Point **F**, however, is not contained in $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{C}\})$ since each of the games v_S, $S \subseteq N$ with $|S| \geq 2$, has an empty core.

Note that **C** is an interior point of the triangle $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{D}\})$. This is due to the fact that each game $-1_{N\setminus\{i\}}$, $i \in N$, can be written as

$$-1_{N\setminus\{i\}} = -1_{\{i\}} + \sum_{S \ni i, 2 \leq |S| \leq |N|-2} (-1_S) + u_i + \hat{w}.$$

In order to emphasize the relative position of **C** in $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{D}\})$ we assert that **C** is not contained in each of the two triangles $\mathrm{Conv}(\{\mathbf{A}, \mathbf{D}, \mathbf{E}\})$ and $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{F}\})$. To demonstrate this let us suppose that $-1_{N\setminus\{i\}} = v_A + v_D + v_E$, for an $i \in N$, $v_A \in \mathbf{A}$, $v_D \in \mathbf{D}$ and $v_E \in \mathbf{E}$. Let $\beta_S \geq 0$, $S \subseteq N$ with $|S| \geq 2$, be such that $v_E = \sum_{S \subseteq N, |S| \geq 2} \beta_S w_S$. Now $-1_{N\setminus\{i\}}$, v_D and v_E are zero-normalized and, therefore, also v_A is zero-normalized. We conclude that $v_A \in \mathrm{Cone}(\{-1_S \colon S \subset N,\ 2 \leq |S| \leq |N| - 2\})$ implying $b^{v_A} = 0$. Then $\mathbb{1}^{\{i\}} = b^{-1_{N\setminus\{i\}}} = b^{v_A} + b^{v_D} + b^{v_E} = b^{v_E} = \sum_{S \subseteq N, |S| \geq 2} \beta_S b^{w_S} = \sum_{S \subseteq N, |S| \geq 2} \beta_S \mathbb{1}^S$ which is impossible. Therefore,

$$-1_{N\setminus\{i\}} \notin \mathrm{Cone}(\{\mathbf{A}, \mathbf{D}, \mathbf{E}\}) = \mathrm{Conv}(\{\mathbf{A}, \mathbf{D}, \mathbf{E}\}).$$

In order to prove that $-1_{N\setminus\{i\}}$ is not contained in $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{F}\})$ we first note that the line segment $\mathrm{Conv}(\{\mathbf{A}, \mathbf{B}\})$ is contained in the cone of games for which the marginal contribution allocation is a core allocation (cf. Derks 1989). Now suppose $-1_{N\setminus\{i\}}$ equals the sum $v_{AB} + v_F$, for an $i \in N$, $v_{AB} \in \mathrm{Conv}(\{\mathbf{A}, \mathbf{B}\})$ and $v_F \in \mathbf{F}$. One easily verifies that the values of the grand coalition and the one-person coalitions equal zero for both games $-1_{N\setminus\{i\}}$ and v_F. Therefore, this holds also for v_{AB} implying that all its core allocations equal zero, in particular $b^{v_{AB}} = 0$. Let $\beta_S \geq 0$, $S \subseteq N$ with $|S| \geq 2$, be such that $v_F = \sum_{S \subseteq N, |S| \geq 2} \beta_S v_S$. Then $\mathbb{1}^{\{i\}} = b^{-1_{N\setminus\{i\}}} = b^{v_{AB}} + b^{v_F} = b^{v_F} = \sum_{S \subseteq N, |S| \geq 2} \beta_S b^{v_S} = \sum_{S \subseteq N, |S| \geq 2} \beta_S \mathbb{1}^S$ which is impossible. We conclude that

$$-1_{N\setminus\{i\}} \notin \text{ Cone}(\{\mathbf{A}, \mathbf{B}, \mathbf{F}\}) = \text{ Conv}(\{\mathbf{A}, \mathbf{B}, \mathbf{F}\}).$$

We assert that all considered classes of games correspond to a convex hull of points in figure 1 as follows:

semibalanced games	$\mathbf{S^N}$	= Conv({**A**,**B**,**D**}),
quasi-balanced games	$\mathbf{Q^N}$	= Conv({**A**,**B**,**F**,**C**}),
balanced games	$\mathbf{B^N}$	= Conv({**A**,**B**,**C**}) and
semiconvex games	$\mathbf{SC^N}$	= Conv({**A**,**C**,**D**,**E**}).

The latter equality is induced by corollary 4. For the proofs of the other equalities the reader is referred to Derks (1989).

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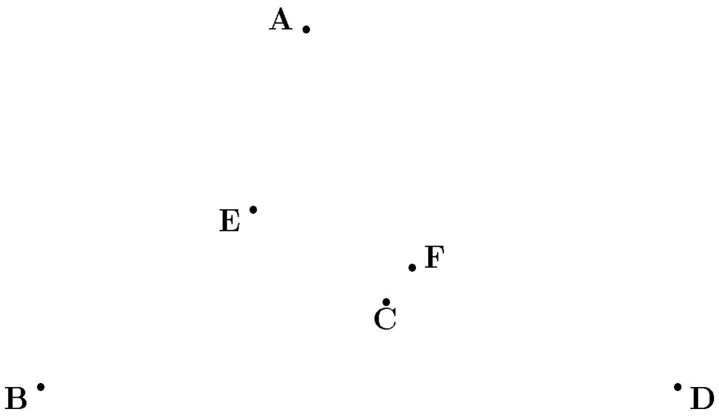

Figure 1.