A new proof for Weber's characterization of the random order values

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Abstract

The paper introduces an alternative proof for Weber's axiomatization of the random order values. In this proof we make use of a combinatorial result that has recently been discussed in Vasil'ev (2002). In addition, we provide a new and intuitive proof for this combinatorial result.

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1 Introduction

One of the main lines of research in cooperative game theory is the search for, and discussion of, intuitive and simple properties for solution functions on the class of transferable utility games. The most well known axiomatic frameworks are those for the single-valued Shapley value (Shapley 1953) and for the set-valued solutions Core and the Weber set, the set of all random order values. The purpose of this paper is to provide a simple and intuitive proof of Weber's result in (1988), stating that the random order values are exactly those solutions fulfilling the properties of efficiency, linearity, monotonicity and the null-player property. Our proof of this result is based on a combinatorial result stated in Vasil'ev (2002), and our main contribution is a new and intuitive proof for this result.

2 Preliminaries

Let N denote a finite set of *players*. We assume that these players are labeled 1 to $n = |N|$: $N = \{1, \ldots, n\}$. A cooperative game with transferable utilities, or simply a *game*, is a real-valued function $v\colon 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$; the value $v(S)$ for each non-empty subset S of N is considered as the payoff that can be achieved if the players in S cooperate. Non-empty subsets of the player set are, therefore, called *coalitions*. The set of all games on the player set N is denoted by V. Further, we denote the collection of all coalitions by $\Omega = \{S \subseteq N \mid S \neq \emptyset\}$.

A *payoff vector* is a vector $x \in \mathbb{R}^n$ assigning payoff $x_i \in \mathbb{R}$ to player $i \in N$. For a payoff vector $x \in \mathbb{R}^n$ and $S \in \Omega$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff to the players in coalition S.

One of the main issues in cooperative game theory is the question how to distribute of the worth $v(N)$ of the grand coalition among the players. A payoff vector x is therefore said to be *efficient* if the total payoff $x(N)$ equals $v(N)$. An efficient payoff vector x, satisfying $x(S) \geq v(S)$ for each coalition S, is called *stable* for obvious reasons. The set of stable payoff vectors is called the *core* of the game v. Unfortunately, the core may be empty.

With Π we denote the set of all permutations $\pi \colon N \to N$ on the player set N. We consider the permutations as line-ups of the players, with $\pi(i)$ denoting the position of player i in the line-up. So, for $\pi \in \Pi$, $\pi^i = \{j \in N | \pi(j) \leq \pi(i)\}$ is the set of all the predecessors of player i, including i herself. Then, the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of a game v and permutation π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad i \in N.$$

It assigns to player i the marginal contribution to the worth of the coalition consisting of all her predecessors in π.

The well known *Shapley value*, introduced by Shapley (1953), has been characterized in many different ways. Perhaps best known is the fact that the Shapley value equals the average of the marginal contribution vectors over all permutations. It is an element of the convex hull of the marginal contribution vectors of v, denoted by $W(v)$, and is called the *Weber set*. Contrary to the core, the Weber set is always non-empty. It contains the core as a subset, as shown by Weber (1988).

The core and the Weber set coincide if and only if the game v fulfills the inequalities $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every pair of coalitions S, $T \in \Omega$ (Shapley, 1971 and Ichiishi, 1981). Examples of games where the core and Weber set coincide are the *unanimity games* u^S, $S \in \Omega$, defined by $u^S(T) = 1$ if $T \supseteq S$, and $u^S(T) = 0$ otherwise.

The *dividends* $\Delta^S(v)$, $S \in \Omega$, of the game v, as defined by Harsanyi (1959, 1963), follow recursively from the system of equations

$$v(S) = \sum_{T \in \Omega : T \subset S} \Delta^T(v), \quad S \in \Omega.$$

It follows that $v = \sum_{S \in \Omega} \Delta^S(v) u^S$, so that each game can be written as a linear combination of the unanimity games. The linear independence of the unanimity games imply that they form a basis for the vector space of characteristic functions; see e.g. Shapley (1953). In the sequel we also need the, in linear spaces more standard, basis of unit vectors, which is formed by the so-called *unity games* $\mathbf{1}^S$, $S \in \Omega$, defined by $\mathbf{1}^S(T) = 1$ if $T = S$ and $\mathbf{1}^S(T) = 0$ for every coalition $T \neq S$.

3 The axiomatization of the random order values

In this section we will provide a new proof of Weber's (1988) main result, which states that the payoff vectors in the convex hull of the marginal contribution vectors, the Weber set, are characterized by four properties. We first recall the properties used by Weber to characterize the payoff vectors in the Weber set.

A function $\theta\colon V \to \mathbb{R}^n$ assigning to each game $v \in V$ a payoff vector $\theta(v)$ is called a *solution*; a solution allocates to each player $i \in N$ her worth or reward $\theta_i(v)$ in the game v. Examples of solutions are the marginal contribution vectors and combinations. Solutions that can be written as a convex combination of marginal contribution vectors, like the Shapley value, are the so called random order values (Weber 1988), i.e., a solution θ is called a *random order value* if a nonnegative weight system $\lambda = (\lambda_\pi)_{\pi\in\Pi}$, satisfying $\sum_{\pi\in\Pi} \lambda_\pi = 1$, exists such that:

$$\theta(v) = \sum_{\pi\in\Pi} \lambda_\pi m^\pi(v), \quad \text{for all games } v \in V.$$

The marginal contribution vectors, and therefore also the random order values, are examples of solutions that distribute a (fixed) weighted combination of the marginal contributions to the player. In order to capture this kind of solution consider a weight system $q = (q_i^S)_{S\in\Omega, i\in S}$ with arbitrary (possibly negative) values q_i^S for $S \in \Omega$ and $i \in S$, and define the following solution

$$\psi_i^q(v) = \sum_{S\ni i} q_i^S(v(S) - v(S \setminus \{i\})), \quad i \in N, \quad v \in V.$$

We call a solution θ to be a *Weber value* if there exists a system q of weights such that $\theta = \psi^q$.

A well established approach to solution theory, initiated by Shapley (1953) with the introduction of the Shapley value, is to pose a set of properties that a solution should reasonably fulfill, and given this framework of properties, to characterize those 'reasonable' solutions. We consider the following properties.

- **ADD** (Additivity): a solution θ is additive if for each two games $v,\ w \in V$ the solution assigned to the sum of these games equals the sum of the solutions of the games: $\theta(v+w) = \theta(v) + \theta(w)$.

- **LIN** (Linearity): a solution θ is called linear if **ADD** holds and $\theta(cv) = c\theta(v)$ for each game v and scalar c.

- **NUL** (Null player property): a solution θ satisfies the null player property if $\theta_i(v) = 0$ for each game $v \in V$ and null player $i \in N$ in v (i is a *null player* in v if $v(S \cup \{i\}) = v(S)$ holds for all coalitions $S \subseteq N \setminus \{i\}$).

- **EFF** (Efficiency): θ is efficient if $\theta(v)(N) = v(N)$ for each game $v \in V$, i.e., the solution distributes the value of the grand coalition among the players.

- **SYM** (Symmetry): a solution θ is symmetric if $\theta_i(v) = \theta_j(v)$ for each game $v \in V$ and symmetric players i and j in v (players i and j are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for all coalitions $S \subseteq N \setminus \{i, j\}$).

- **MON** (Monotonicity): a solution θ satisfies the monotonicity property if for each game $v \in V$ and each monotonic player $i \in N$ we have $\theta_i(v) \geq 0$ (a player i is *monotonic* whenever $v(S \cup \{i\}) \geq v(S)$ for each $S \subseteq N \setminus \{i\}$).

- **POS** (Positivity): a solution θ is positive if $\theta(v) \geq 0$ for each unanimity game v.

It is well known that the Shapley value is the unique solution that satisfies **ADD**, **NUL**, **EFF**, and **SYM** (Shapley 1953). Further, it should be observed that each random order value satisfies efficiency, but that a Weber value does not need to be efficient. More specifically, the following results may be found in Weber (1988) (with a slightly different **NUL**-property, but equivalent with the present one in the context of **LIN**).

Lemma 1 *(Weber 1988)*

1) *A solution satisfies* **LIN** *and* **NUL** *if and only if it is a Weber value.*

2) *A Weber value satisfies* **MON** *if and only if the corresponding sharing system q is non-negative, i.e., $q_i^S \geq 0$ for each $S \in \Omega$ and $i \in S$.*

3) *A Weber value satisfies* **EFF** *if and only if the corresponding sharing system q fulfills*

$$q^N(N) = 1 \text{ and } q^S(S) = \textstyle\sum_{j \notin S} q_j^{S \cup \{j\}} \text{ for each } S \in \Omega,\ S \neq N. \qquad (1)$$

(as usual, $q^S(S)$ is an abbreviation for $\sum_{i \in S} q_i^S$)

Let Q^* be defined as the set of non-negative valued weight systems $(q_i^S)_{S \in \Omega, i \in S}$, that fulfill (1):

$$Q^* = \{q = (q_i^S)_{S \in \Omega, i \in S} \mid q \geq 0,\ q^N(N) = 1,\ q^S(S) = \sum_{j \notin S} q_j^{S \cup \{j\}},\ S \in \Omega \setminus \{N\}\},$$

Using the set of weight systems Q^*, Lemma 1 can be summarized shortly as follows.

Corollary 2

A solution $\theta\colon V \to \mathbb{R}^N$ satisfies **LIN**, **NUL**, **EFF** *and* **MON** *if and only if there exists $q \in Q^*$ such that $\theta = \psi^q$.*

It is easily shown that the marginal contribution vectors fulfill all four properties, mentioned in the corollary. For $\pi \in \Pi$, let m^π be the solution assigning the marginal vector $m^\pi(v)$ to a game $v \in V$. Then the corollary says that there exists a weight system $\widehat{q} \in Q^*$ such that $\psi^{\widehat{q}} = m^\pi$. Indeed, for $\pi \in \Pi$, let the weight system q^π be given by

$$(q^\pi)_i^S = \begin{cases} 1 \;\text{ if }\; S = \pi^i, \\ 0 \;\text{ otherwise.} \end{cases} \tag{2}$$

Clearly, for an arbitrary game $v \in V$ and player $i \in N$, it follows that

$$\psi_i^{q^\pi}(v) = \sum_{S \ni i} (q^\pi)_i^S (v(S) - v(S \setminus \{i\})) = (v(\pi^i) - v(\pi^i \setminus \{i\})) = m_i^\pi(v), \tag{3}$$

showing that the weight system q^π induces the corresponding marginal vector, i.e. $\psi^{q^\pi}(v) = m^\pi(v)$ for each game $v \in V$. We saw already in the proof of statement 1) of Lemma 1 that for a Weber value ψ^q we have $\psi_i^q(\mathbf{1}^S) = q_i^S$, for $S \in \Omega$ and $i \in S$, so that the value uniquely determines the corresponding weight system. This showes that $q^\pi = \widehat{q} \in Q^*$.

It is first shown in Vasil'ev (2002) that the elements q^π are the extreme points of Q^* (see Vasil'ev and van der Laan (2002) for an alternative and shorter approach, in English).

We will provide a direct and intuitive proof of Vasil'ev's combinatorial result. To this end, observe that the set Q^* is a polyhedron of the following form

$$F(A, b) = \{x \in \mathbb{R}_+^k \mid Ax = b\},$$

for a vector $b \in \mathbb{R}^m$ and real-valued $(m \times k)$-matrix A. With elementary arguments one shows that an extreme element of $F(A, b)$ can be recognized by its carrier (where the *carrier* of a vector x is defined by $C(x) = \{i \mid x_i \neq 0\}$):

Lemma 3

An element x of $F(A, b)$ is extreme if and only if its carrier is minimal, i.e., there is no $y \in F(A, b)$, $y \neq x$, such that $C(y) \subseteq C(x)$.

This result is not unknown within the settings of cooperative game theory: it has been applied, although implicitly, by Shapley in (1967) for proving that only the minimal balanced collections are essential in the characterization of the games with a non-empty core.

Theorem 4 *(Vasil'ev 2002)*

The set Q^* is the convex hull of its extreme elements q^π, $\pi \in \Pi$.

Proof : We showed already that the weight systems q^π, $\pi \in \Pi$, have to be elements of Q^*. We will prove now that these elements are the only candidates for being extreme in Q^*. By Theorem 3 this is true if we can show that for each $q \in Q^*$ there is a permutation π such that $C(q^\pi) \subseteq C(q)$. So, let $q \in Q^*$ be arbitrary. We will construct a sequence of players $i_1, i_2, \ldots, i_n$ and sets $S_1, S_2, \ldots, S_n$ such that $S_1 = N$, $i_k \in S_k$, and $S_k = S_{k-1} \setminus \{i_{k-1}\}$, $k = 2, \ldots, n$, by the following procedure.

> **Step 1.** Take $S_1 = N$. From $q^N(N) = 1$, it follows that there exists a player $i_1 \in S_1$ with $q_{i_1}^{S_1} > 0$. Let $k = 2$.
>
> **Step 2.** Define $S_k = S_{k-1} \setminus \{i_{k-1}\}$. Then, $q^{S_k}(S_k) = \sum_{j \notin S_k} q_j^{S_k \cup \{j\}} \geq q_{i_{k-1}}^{S_k \cup \{i_{k-1}\}} = q_{i_{k-1}}^{S_{k-1}} > 0$. This implies that there exists a player $i_k \in S_k$ with $q_{i_k}^{S_k} > 0$.

Step 3 To obtain $i_3, \ldots, i_n$ and $S_3, \ldots, S_n$, repeat Step 2 for $k = 3, \ldots, n$.

Observe that $S_n = \{i_n\}$ and $q_{i_k}^{S_k} > 0$ for all $k = 1, \ldots, n$. Define the permutation of players $\pi \in \Pi$ by $\pi = (i_n, i_{n-1}, \ldots, i_2, i_1)$. Then, according to formula (2), we have $(q^\pi)_i^S = 1$ if and only if $S = S_k$ and $i = i_k$, $k = 1, \ldots, n$, and $(q^\pi)_j^S = 0$ otherwise. It is evident that $C(q^\pi) \subseteq C(q)$ holds.

The total number of positive coordinates of q^π equals n, and different permutations lead to different carriers for the corresponding weight systems. We may therefore conclude that the elements q^π, $\pi \in \Pi$, all are extreme.

A bounded convex polyhedron is equal to the convex hull of its extreme elements. We are, therefore, ready with the proof when we have shown that Q^* is bounded. Suppose therefore, that Q^* is unbounded. Then, due to the fact that Q^* is a polyhedron, containing only non-negative elements, there exist two elements $q, \widehat{q} \in Q^*$, $q \neq \widehat{q}$, such that $q \geq \widehat{q}$. Observe that $q^N(N) = \widehat{q}^N(N)$ so that from $q^N \geq \widehat{q}^N$ we conclude that $q^N = \widehat{q}^N$. Now suppose that $q^T = \widehat{q}^T$ for all $T \subseteq N$ with cardinality greater than m, and let S be a coalition with $|S| = m$. We have

$$q^S(S) = \sum_{k \notin S} q_k^{S \cup \{k\}} = \sum_{k \notin S} \widehat{q}_k^{S \cup \{k\}} = \widehat{q}^S(S).$$

From $q^S \geq \widehat{q}^S$ we thus derive $q^S = \widehat{q}^S$, so that by induction we conclude that $q = \widehat{q}$, a contradiction. This shows the boundedness of Q^*. □

Observe that the spanning elements q^π, $\pi \in \Pi$, of Q^* fulfill $\sum_{S \ni i} (q^\pi)_i^S = 1$ for each $i \in N$. Therefore, this has to hold for all $q \in Q^*$ and $i \in N$:

$$\sum_{S \ni i} q_i^S = 1.$$

In particular, $q_i^S \leq 1$ for all $q \in Q^*$, $S \in \Omega$ and $i \in S$.

We are now able to provide an alternative proof for Weber's characterization of the random order values. First, observe that we obtained the following description of the Weber set, with $v \in V$:

$$W(v) = \{\psi^q(v) \mid q \in Q^*\}.$$

For a discussion and an application in the context of Harsanyi payoff vectors the reader is referred to Derks *et al.* (2002) See also the conclusion of this section.

Theorem 5 *(Weber 1988)*

A solution $\theta\colon V \to \mathbb{R}^n$ *satisfies* **LIN**, **NUL**, **EFF** *and* **MON** *if and only if* θ *is a random order value.*

Proof: Let θ be a solution, satisfying **LIN**, **NUL**, **EFF** and **MON**. According to Corollary 2 there exists a weight system $q \in Q^*$ so that $\theta = \psi^q$. Applying Theorem 4, there are non-negative weights $(\lambda_\pi)_{\pi\in\Pi}$, satisfying $\sum_{\pi\in\Pi} \lambda_\pi = 1$ such that $q = \sum_{\pi\in\Pi} \lambda_\pi q^\pi$. Observe now that for each game v and player i we have

$$\begin{aligned}
\psi_i^q(v) &= \sum_{S\ni i} \sum_{\pi\in\Pi} \lambda_\pi (q^\pi)_i^S (v(S) - v(S \setminus \{i\})) \\
&= \sum_{\pi\in\Pi} \lambda_\pi \sum_{S\ni i} (q^\pi)_i^S (v(S) - v(S \setminus \{i\})) \\
&= \sum_{\pi\in\Pi} \lambda_\pi \psi_i^{q^\pi}(v) \\
&= \sum_{\pi\in\Pi} \lambda_\pi m_i^\pi(v),
\end{aligned}$$

where the last equality follows from equation (3). This shows that θ is a random order value.

In the same way one shows that for a random order value $\theta = \sum_{\pi\in\Pi} \lambda_\pi m_i^\pi(v)$, with non-negative weights $(\lambda_\pi)_{\pi\in\Pi}$, satisfying $\sum_{\pi\in\Pi} \lambda_\pi = 1$, we have $\theta = \psi^q$, with $q = \sum_{\pi\in\Pi} \lambda_\pi q^\pi \in Q^*$. Corollary 2 now implies that θ fulfills **LIN**, **NUL**, **EFF** and **MON**. □

The proof of Theorem 5 in Weber (1988) is based on a specification of the interrelation between the two different weight systems q and λ. It is not hard to show that for a weight system $\lambda = (\lambda_\pi)_{\pi\in\Pi}$ with non-negative weights, and $\sum_{\pi\in\Pi} \lambda_\pi = 1$, the corresponding weight system $q = (q_i^S)_{S\in\Omega, i\in S}$ defined by

$$q_i^S = \sum_{\pi\in\Pi:\pi^i=S} \lambda_\pi, \quad S \in \Omega,\ i \in S,$$

is an element of Q^*, and $\psi^q = \sum_{\pi\in\Pi} \lambda_\pi m^\pi$.

Specifying λ in terms of the q-weights is more complex, and the corresponding proofs that the λ-weights fulfill the necessary properties, follow some repeated summation and induction techniques. To our knowledge there is no alternative proof available in existing literature.

Call a solution θ a *Harsanyi value* if there exists a non-negative weight system $p = (p_i^S)_{S\in\Omega, i\in S}$, with $p^S(S) = 1$ for all $S \in \Omega$, such that

$$\theta(v) = \phi^p(v) = \sum_{S\ni i} p_i^S \Delta^S(v), \quad \text{for all } v \in V,$$

i.e., a Harsanyi value distributes the dividends of the coalitions among the coalition members according to some non-symmetric (probability) weights. In Vasil'ev (1988) and Derks *et al.* (2000) it is shown that a solution θ on V satisfies **LIN**, **NUL**, **EFF** and **POS** if and only if θ is a Harsanyi value. Since

MON is stronger than **POS**, this implies that each random order value is a Harsanyi value. This essentially shows that the Weber set is a subset of the set of Harsanyi values, for each game. For a further discussion on this inclusion result, and, in particular, the set of Harsanyi values, see Derks *et al.* (2002), and the references therein.

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