On the Number of Extreme Points of the Core of a Transferable Utility Game *

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Abstract

We prove that the number of extreme points of the polyhedral sets of the form $P_{A,b} = \{x \in I\!R^n \,|\, Ax \geq b\}$, with A an integer valued matrix, is bounded from above by $n!$ times the volume of the convex hull of the zero vector and the rows of A. This result is applied on 0,1-valued matrices, obtaining an upper bound of $n!$ for the number of extreme points. This implies that the upper core and the core of a transferable utility game have at most $n!$ different extreme points, with n the number of players. The maximum number is attained by strict convex games but other games may have this property as well. These games have to be strict upper exact, must have a large core and fulfill a kind of non-degeneratedness, and since these properties imply strict exactness, $n!$ different extreme core points can only be obtained by strict exact games. We show that not all games with these properties have $n!$ different extreme points.

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1 Introduction

Stability of an allocation among a group of players is normally considered to refer to the property that there is no incentive among subgroups or coalitions of players to deviate from the given allocation and choose the alternative of cooperation. In a transferable utility game the stable allocations are exactly the elements of the upper core. These allocations always exist but may not be feasible in the sense that the total payoff exceeds the total earnings of the grand coalition. The core of a game is the set of feasible allocations within the upper core. It is a (possibly empty) face of the upper core.

The core is perhaps the best known solution concept within Cooperative Game Theory. The first contributions within this context are found in Gillies (1953). It is generally believed that the core and core-like structured sets have at most $n!$ extreme points. This is indeed the case and the main contribution of this note is to provide a proof.

With core-like structured sets we denote those sets that can appear as a core of a game. Examples are the so-called core covers, which are generalizations of the core, and are introduced mainly in order to bypass the dissatisfactory property of the core that it may be empty. The first results in this direction are found in Tijs (1981) and Tijs and Lipperts (1982). Other examples of core-structures are the anti-core, the least core and the Selectope. Vasilev (1981) and, recently, Derks, Haller and Peters (2000) are contributions dealing with the core structure of the Selectope.

Although our first concern is the core, the main results and concepts deal with the upper core. The upper core of a game can be described as the feasible region of a suitably chosen linear program, where the matrix is 0,1-valued and the constraint vector coefficients are the values of the coalitions. Actually, we are dealing with polyhedra of the type $P_{A,b} = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where A is an $m \times n$ integer valued matrix and $b \in \mathbb{R}^m$. In the literature there is a comprehensive study on the upper bound on the number of extreme points of such polyhedra. It is well-known that $x \in P_{A,b}$ is an extreme point of $P_{A,b}$ if and only if there is a set of n linearly independent vectors among the rows a of A for which the equality $a\,x = b_a$ (with b_a the coefficient of b associated with row a) holds. Hence, a trivial upper bound for the number of extreme points of $P_{A,b}$ is $\binom{m}{n}$. McMullen (1970) showed that this is an overestimate and he proved that the polyhedron $P_{A,b}$ has at most

$$f(m,n) = \binom{m-\lfloor\frac{n+1}{2}\rfloor}{m-n} + \binom{m-\lfloor\frac{n+2}{2}\rfloor}{m-n}$$

extreme points. Furthermore, Gale (1963) constructed examples of polyhedra having precisely $f(m,n)$ extreme points, so that McMullen's bound cannot be further improved for arbitrary matrices A (see also Chvátal (1983) for these results).

Our main result is that for polyhedra $P_{A,b}$, where A is an $m \times n$-matrix with 0,1-valued coefficients, an upper bound of $n!$ different extreme points exists. This is an improvement of McMullen's upper bound in case A contains all possible 0,1-valued row vectors: $f(2^n, n) \geq 2(\frac{n}{2})^n \geq n!$, and by the Stirling approximation of $n!$, being approximately $\sqrt{(\pi n)}(\frac{n}{e})^n$, we observe that the McMullen upper bound exponentially exceeds the value $n!$.

The proof of our main result is based on a polar version of an argument, stated by Imre Bárány (see Ziegler (1995), page 25) in the context of the related search for upper bounds for the number of facets of a 0/1-polytope, which is defined to be the convex hull of a set of elements with 0,1-valued coefficients. This argument shows that $n! + 2n$ is an upper bound of the number of facets of any n-dimensional 0/1-polytope. The polar translation of a 0/1-polytope is a polyhedron of type $-P_{A,-\mathbf{1}}$, with A an $m \times n$-matrix with 0,1-valued coefficients, and $\mathbf{1}$ the m-vector $(1, \ldots, 1)$, so that for these polyhedra Bárány's upper bound directly implies a maximum number of $n! + 2n$ extreme points. With a bit more effort we will obtain a better upper bound for the larger class of polyhedra where the constraint vector b may admit arbitrary values.

Let us define the polytope Q_A as the convex hull of the origin 0 and all row vectors of the matrix A. In section 2 we shall prove that the number of extreme points of $P_{A,b}$ is bounded by $n!$ times the n-dimensional volume of the polytope Q_A. Since Q_A is contained in the unit hypercube, described by the restrictions $0 \leq x_i \leq 1$ for all $i \in \{1, \ldots, n\}$ if A is a 0,1-valued matrix, Q_A has a volume of at most 1, and it follows that the polyhedron $P_{A,b}$ has at most $n!$ extreme points.

In section 3 we formally introduce the cooperative game model, and state that the core, being a face of a polyhedron of type $P_{A,b}$, with A a (0,1)-matrix, has at most $n!$ different extreme core points. Strict convexity implies that the core actually has $n!$ different extreme points, but we show that there are more games with this property. We further discuss an intuitive and direct approach for listing $n!$ extreme points of the core (possibly with duplicates) but we show that this approach may fail to list all extreme points, thus showing that it cannot be used for establishing a maximum on the number of extreme points.

Section 4 describes some properties that are induced by having $n!$ extreme core points. These are the large core property, a kind of strict exactness, and a non-degeneracy property. We supply an example that these properties are not sufficient for obtaining $n!$ extreme core points.

In section 5 we conclude the paper with a summary.

2 Main Results

Let $X \subseteq I\!\!R^n$ and $x \in X$. The vector x is said to be an *interior* point of X if there exists $\epsilon > 0$ such that for all $d \in I\!\!R^n$ with $||d|| \leq 1$ and all δ with $0 \leq \delta \leq \epsilon$ we have $x + \delta d \in X$. Here $||d||$ denotes the Euclidian norm of the vector d. The set of all interior points of X is called the *interior* of X.

For any two vectors $x, y \in I\!\!R^n$ their inner product is denoted by $x\,y$. We shall denote the righthand side associated with row a of matrix A by b_a.

For $x \in P_{A,b}$ let I_x denote the set of rows a for which $a\,x = b_a$. Further, let $Q_x \subseteq Q_A$ denote the convex hull of 0 and the vectors of I_x. It is intuitively quite clear that $Q_x \cap Q_y$ has an empty interior for any two distinct extreme points $x, y \in P_{A,b}$. For the sake of completeness we provide a proof here.

Lemma 1 *Let x and y be two distinct extreme points of $P_{A,b}$. Then $Q_x \cap Q_y$ has an empty interior.*

Proof : Let x and y be two distinct extreme points of $P_{A,b}$ and suppose that $Q_x \cap Q_y$ has a non-empty interior. Choose z in the interior of $Q_x \cap Q_y$. So, z lies also in the interior of Q_x and therefore, it can be written as a convex combination of the extreme points of the convex set Q_x with all coefficients strictly positive, i.e. $z = \sum_{a \in I_x} \lambda_a a$ with $\lambda_a > 0$. We have that $y\,\hat{a} > b_{\hat{a}}$ for at least one $\hat{a} \in I_x$. Since $\lambda_{\hat{a}} > 0$, it follows that $zy = \sum_{a \in I_x} \lambda_a\, ay > \sum_{a \in I_x} \lambda_a ax = zx$. Analogously one proves that $zx > zy$, a contradiction. $\square$

As a consequence of lemma 1, the n-dimensional volume of the union of polytopes Q_x is simply the sum of their volumes. In the following we shall provide a lower bound on the volume of Q_x. This gives us then an upper bound on the number of polytopes Q_x that can be contained in Q_A, or equivalently, it gives us an upper bound on the number of extreme points of $P_{A,b}$.

Let us denote the volume of an n-dimensional body $X \subseteq I\!R^n$ by $\mathcal{V}(X)$. The following theorem is well-known in linear algebra. For a proof we refer to Birkhoff and MacLane (1963).

Theorem 2 *Let $X \subseteq I\!R^n$ be a $\mathcal{V}$-measurable set, and let A be a square matrix of dimension n. Then $\mathcal{V}(\{Ax \mid x \in X\}) = |\det(A)|\, \mathcal{V}(X)$, where* $\det(A)$ *denotes the determinant of the matrix A.*

Now we are in a position to provide a lower bound on $\mathcal{V}(Q_x)$.

Lemma 3 *Let A be an $m \times n$ integer valued matrix and let $b \in I\!R^n$. Furthermore, let x be an extreme point of $P_{A,b}$. Then $\mathcal{V}(Q_x) \geq 1/n!$.*

Proof: Since x is an extreme point of $P_{A,b}$, I_x contains a set of n independent vectors, say $x^1, x^2, \ldots, x^n$. According to Theorem 2 the volume of the convex hull of the points $x^1, x^2, \ldots, x^n$ and 0 equals $|\det(x^1, \ldots, x^n)|\mathcal{V}(\{\lambda \in I\!R^n_+ : \sum_{i=1}^n \lambda_i \leq 1\})$, where $\det(x^1, \ldots, x^n)$ denotes the determinant of the matrix with columns $x^1, \ldots, x^n$. All entries of this matrix are integer, so its determinant is also integer. The independent nature of the columns in the matrix ensures that the determinant is unequal to 0 and therefore, $|\det(x^1, \ldots, x^n)| \geq 1$. Consequently, the volume of the convex hull of the points $x^1, x^2, \ldots, x^n$ and 0 is at least the volume of $\{\lambda \in I\!R^n_+ : \sum_{i=1}^n \lambda_i \leq 1\})$, which equals $1/n!$. Since this convex hull is contained in Q_x, also Q_x has a volume of at least $1/n!$. □

Observe that the lower bound of $1/n!$ can only be achieved by $\mathcal{V}(Q_x)$ if and only if there is a set of n independent vectors $x^1, x^2, \ldots, x^n$ in Q_x with $|\det(x^1, \ldots, x^n)| = 1$, and there are no elements of Q_x outside the convex hull of $x^1, x^2, \ldots, x^n$ and 0.

Theorem 4 *Let A be an $m \times n$ integer valued matrix and let $b \in I\!R^n$. Then $P_{A,b}$ has at most $n!\mathcal{V}(Q_A)$ extreme points.*

Proof : Let E denote the set of extreme points of $P_{A,b}$. Clearly, $Q_x \subseteq Q_A$ for all $x \in E$. Hence, $\cup_{x \in E} Q_x \subseteq Q_A$ and

$$\mathcal{V}(Q_A) \geq \mathcal{V}(\cup_{x \in E} Q_x).$$

According to lemma 1, the intersection of any two polytopes Q_x and Q_y with $x, y \in E$ $(x \neq y)$ has an empty interior, and therefore

$$\mathcal{V}(\cup_{x \in E} Q_x) = \sum_{x \in E} \mathcal{V}(Q_x).$$

Furthermore, each polytope Q_x has a volume of at least $1/n!$. Hence,

$$\sum_{x \in E} \mathcal{V}(Q_x) \geq |E|/n!.$$

Combining these results the theorem follows. □

Corollary 5 *For any $m \times n$ (0,1)-matrix A and m-vector b the polyhedron $P_{A,b}$ has at most $n!$ extreme points.*

The maximum of $n!$ can only be achieved if every $(0, 1)$-vector except the null vector is a row of A. Clearly, if A has less rows, then $\mathcal{V}(Q_A)$ is strictly less than 1, and hence the bound in theorem 4 is strictly less than $n!$. If not every $(0, 1)$-vector is a row of the matrix A we therefore obtain a stronger bound.

The maximum of $n!$ extreme points of $P_{A,b}$ can actually be achieved, with A chosen 'maximal' as indicated. Examples are given e.g. in Edmonds (1970), and in Shapley (1971) in the context of transferable utility games.

To obtain $n!$ different extreme points in $P_{A,b}$ the sets Q_x, being 0/1-polytopes, should all have volume $1/n!$ for each extreme point x of $P_{A,b}$, and this is only possible if Q_x is a simplex, i.e., the convex hull of a set of $n + 1$ affine independent vectors (see the observation following the proof of Lemma 3). Further, the union of these simplices should coincide with the unit hypercube. This gives rise to a simplicial subdivision of the unit hypercube, also called a triangulation. The main issue in the literature on triangulations is the minimal number of simplices needed to form a triangulation (Mara (1976), Hughes (1994)). The so-called standard triangulation is the subdivision of the hypercube in simplices of the form $Q^{\pi} = \{x \in I\!R^n : x_{\pi(1)} \geq$

$x_{\pi(2)} \geq \ldots x_{\pi(n)} \geq 0,\ \sum_{i=1}^{n} x_i = 1\}$, with π running over all permutations on $\{1, 2, \ldots, n\}$. See Freudenthal (1942) for an early reference (Todd 1976, pp 29—30). The standard triangulation pops up in many situations. The next section will provide examples.

3 The Core of a transferable utility Game

An n-person transferable utility game (or *game* for short), with $n \in I\!N$, is a real valued map on the set of subsets of the *player set* $N = \{1, 2, \ldots, n\}$, the empty set excluded. A non-empty subset S of N is referred to as a *coalition*, and its *value* $v(S)$ in the game v is interpreted as the net gain of the cooperation of the players in S.

A game v is said to be *additive* if each coalition value is obtained by summing up the one-person coalition values: $v(S) = \sum_{i \in S} v(\{i\})$ for all coalitions S. Given an *allocation* $x \in I\!R^N$ the corresponding additive game is the game, also denoted by x, with coalition values $x(S) = \sum_{i \in S} x_i$. An allocation x is called *stable* in the game v if the corresponding additive game majorizes v: $x(S) \geq v(S)$ for all coalitions S (or $x \geq v$ for short).

The *upper core* of a game v, denoted UCore(v), is the set of stable allocations. Its elements are interpreted as those payoffs to the players that are preferred to playing the game. However, not all stable allocations are feasible in the sense that they can be afforded by the players. Here, we assume that an allocation x is *feasible* in the game v if $v(N) \geq \sum_{i \in N} x_i$ holds. The *core* of a game v, denoted Core(v), is defined as the set of stable and feasible allocations of v.

Consider a fixed sequence $S_1, S_2, \ldots, S_{2^n-1}$ of the coalitions in N. Let A denote the $(2^n - 1) \times n$-dimensional matrix with j, i-coefficient equal to 1 if player i is a member of coalition S_j, and 0 otherwise. For a game v the upper core obviously equals P_{A,b^v}, with b^v the $(2^n - 1)$-dimensional constraint vector with j-coefficient $v(S_j)$. For a stable allocation x the set Q_x is the convex hull of the zero vector and the indicator functions, the rows of A, corresponding to coalitions S for which equality $x(S) = v(S)$ holds. We will refer to these coalitions as being *tight* .

Feasibility of a stable allocation of course imply that the grand coalition N has to be tight. Therefore, the core of v is equal to the face of the upper core P_{A,b^v} of v, determined by the constraint corresponding to coalition N.

With the help of Corollary 5 we conclude that

Corollary 6 *The core of an n-person cooperative game has at most $n!$ extreme points.*

We will first show that there is a large class of n-person games for which the number of different core points equals the maximum possible number of $n!$. For this we need the following. A game $v : 2^N \to \mathbb{R}$ is called *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all coalitions } S, T \tag{1}$$

(with the convention that the game value of the empty set equals 0). The game is called *strict convex* if the convexity inequalities hold, and none of them with equality whenever $S \not\subseteq T$ or $T \not\subseteq S$.

It is well known that the extreme points of the core of a convex game are among the so called marginal contribution vectors (see Shapley (1971), and Ichiishi (1981) for the converse statement). For a permutation π on the player set N the *marginal contribution* allocation $m(v, \pi)$ in the game v is defined by

$$m_i(v, \pi) = v(P_\pi(i)) - v(P_\pi(i) \backslash \{i\}), \; i \in N,$$

with $P_\pi(i)$ denoting the predecessors j of player i in π: $\pi(j) \leq \pi(i)$. The allocation $m(v, \pi)$ is the final outcome of the procedure where the players enter a room one by one in the order given by π and each player obtains the value of the coalition of players in the room minus what already has been allocated.

Some of the $n!$ marginal contribution allocation vectors may coincide, but if the game is strict convex then all these allocations are different. To show this, first observe that $m(v, \pi)(P_\pi(i)) = v(P_\pi(i))$ for all permutations π and players i, and games v not necessarily (strict) convex. Now let v be a strict convex game, π a permutation of N, and S a coalition unequal to any predecessor set of π. We will prove that $m(v, \pi)(S) > v(S)$. Let σ be a permutation such that the players in S are the first and such that if $\pi(i) < \pi(j)$ for $i, j \in S$ then also $\sigma(i) < \sigma(j)$. A permutation with this property is constructed, for example, by interchanging the positions (in π) of any player $j \notin S$ with a player $i \in S$ with $\pi(j) < \pi(i)$ until there are no players left with this property. For each $i \in S$ we have $P_\sigma(i) \subseteq P_\pi(i)$ so that by applying (1), with $S = P_\sigma(i)$ and $T = P_\pi(i) \backslash \{i\}$, we obtain

$$m_i(v, \sigma) = v(P_\sigma(i)) - v(P_\sigma(i) \backslash \{i\}) \leq v(P_\pi(i)) - v(P_\pi(i) \backslash \{i\}) = m_i(v, \pi). \tag{2}$$

Therefore, $m(v,\pi)(S) \geq m(v,\sigma)(S)$, and since S is a predecessor set of σ it follows that $m(v,\pi)(S) \geq v(S)$ (thus proving that the marginal contribution allocations are elements of the upper core). There is an $i \in S$ such that $P_\sigma(i)$ is a proper subset of $P_\pi(i)$, and for this player strict inequality holds in (2) because of strict convexity of the game. Therefore, $m(v,\pi)(S) > v(S)$ for all coalitions S except the predecessor sets of π. From this, one easily derives that there are no two marginal contribution allocations equal to each other. Consequently, the core has precisely $n!$ extreme points (Shapley (1971)). This shows that the bound in corollary 6 is sharp.

Let us call a collection of coalitions of $\{1, 2, \ldots, n\}$ *regular* if the indicator functions span $I\!R^n$. It is evident that a stable allocation of a game is extreme in the upper core if and only if its tight coalitions form a regular collection, and a stable allocation is an extreme core point if and only if N is tight, and the set of its tight coalitions is regular.

Observe that we actually proved that the tight coalitions of the marginal contribution allocation $m(v,\pi)$, with v strict convex, are precisely its predecessor sets, so that the tight coalitions form a regular collection. The corresponding set $Q_{m(v,\pi)}$, the convex hull of the zero vector and the indicator functions of the predecessor sets of π, is easily seen to equal $Q^\pi = \{x \in I\!R^n : x_{\pi(1)} \geq x_{\pi(2)} \geq \ldots x_{\pi(n)} \geq 0, \sum_{i=1}^n x_i = 1\}$, a typical simplex of the standard triangulation of the unit hypercube. On the other hand, if the tight coalitions of a stable allocation give rise to a simplex of the form Q^π then the allocation has to be a marginal contribution allocation. Therefore, the strict convex games are exactly the games that give rise to the standard triangulation.

The strict convex games are not the only games with core having the maximum number of extreme points as the following example shows.

Consider the non-convex (symmetric) 4-player game v' with values 0,7,12,22 for coalitions with number of players respectively, 1, 2, 3, and 4:

$\lvert S\rvert$:	1	2	3	4
$v'(S)$:	0	7	12	22

Consider the allocations $(2, 5, 5, 10)$ and $(0, 7, 7, 8)$. Obviously, both belong to the core of v', with tight coalition sets respectively $\{N, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ and $\{N, \{1\}, \{1,2\}, \{1,3\}\}$. The two collections are regular, so that we may conclude that the two allocations are extreme in the core. Because of the symmetry among the players in the game any of the 12 allocations with coefficients 2,5,5,10, and the 12 allocations with coefficients 0,7,7,8 are extreme core points. Therefore the game has at least 24 extreme core points. There are no other since 24 is the maximum number: $4! = 24$.

There is an intuitive approach for obtaining the extreme core points: First, take any ordering of the players. Then, take the first player and maximize its pay off among the core allocations. Thereafter, take the next player, and maximize his pay off among the core allocations where the first player gets his maximal pay-off. Continue in this way until the last player. Following this way we obtain an extreme point of the core. Since there are $n!$ different orderings of the players we obtain $n!$ extreme points (possibly, there may be duplicates).

The above example, however, shows that we may not obtain all extreme points in this way. Observe that if we maximize the pay off to a player among the core allocations we obtain the value 10, and therefore, we will never end up in an extreme core allocation with coefficients 0,7,7,8. Analogously, if we minimize the pay-off, instead of maximize, we will not terminate in a core allocation with coefficients 2,5,5,10.

4 Strict exact games

It is not only of mathematical interest to provide necessary and sufficient conditions for games having the maximum number of extreme core points. One may, for example, argue that the extreme points of the core are precisely the outcomes of a game where the players choose their actions in an extreme social way. The number of extreme core points may as such serve as a measure for social complexity (whatever these terms may indicate in an appropriate context). Also, procedures or protocols that construct or give rise to core allocations may endure a complexity that is dependent on the number of extreme core points, especially when, depending on the settings, any core point may occur as outcome.

It is therefore of interest to deduce properties that are implied by the fact that the number of extreme core points is maximal. First, one can easily verify that the number of tight coalitions in an extreme (upper) core allocation should not exceed the dimension of the allocation space. This means that the indicator functions of the tight coalitions are linearly independent. Collections of coalitions with this property are called *non-degenerate*, and a game is called *non-degenerate* if the collections of tight coalitions is non-degenerate for each extreme upper core allocation.

To obtain the maximum number of extreme core points in an n-person game v, the upper core should not have extreme points outside the face corresponding to the grand coalition. This is equivalent to the upper core being equal to the core and all the points lying above the core: $\mathrm{UCore}(v) = \mathrm{Core}(v) + \mathbb{R}^n_+$. If this holds we say that v has a *large core*. A game v has a large core if and only if for each stable allocation x there is a core allocation x' such that $x' \leq x$.

For the upper core having the maximum amount of different extreme points it is essential that in its description as a polyhedral set $P_{A,b}$ no rows of A can be deleted (see remark following Corollary 5). This hints to the condition that for each coalition there is a corresponding face in the upper core, which has to be of maximal dimension $n-1$, a facet. In other words, for each coalition S there is a stable allocation for which S is the only tight coalition. If this is the case then the game is called *strict upper exact*. Without going in detail, it is not hard to prove that strict upper exactness is equivalent to the property that each subgame has a core of maximal dimension.

Proposition 7 *If the core of a game has the maximum of $n!$ different extreme core points then the core has to be large and the game has to be non-degenerate and strict upper exact.*

The next example shows that the converse does not hold. Consider the 5-person game v'' defined by

$\|S\|$:	1	2	3	4	5
$v''(S)$:	0	1	8	11	23

We will show that the extreme stable allocations in the upper core of v'' are the following points:
1) the 20 allocations with coefficients 0,4,4,4,11,
2) the 20 allocations with coefficients 2,3,3,3,12, and

3) the 60 allocations with coefficients 0,1,7,7,8.
It is left to the reader to check the stability property of these allocations, 100 in total (and less than the maximum possible of 5!=120). Also, with the help of these allocations one easily derives that the game is strict upper exact.

The tight coalitions of the stable allocation $(0, 4, 4, 4, 11)$ are the player set $N = \{1, 2, 3, 4, 5\}$, $\{1\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$. The independence of the corresponding indicator functions follows from determining the determinant value of the matrix consisting of these indicator functions, say in the given order. The value equals -2, so that we may conclude that $(0, 4, 4, 4, 11)$ is extreme in the upper core, and due to the symmetry among the players in the game the other 19 allocations with the same coefficients are also extreme in the upper core. Further, the computed determinant value implies that the volume of $Q_{(0,4,4,4,11)}$, the convex hull of the zero vector and the indicator functions of the tight coalitions, equals 2/5!, so that the 20 allocations of type **1)** consume $20 \cdot 2/5!$ of the available volume of the unit hypercube, implying that the upper core can have at most $20 + 120 - 40 = 100$ extreme points.

The other two types of allocations is done in the same way. The tight coalitions of the stable allocation $(2, 3, 3, 3, 12)$ are N, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$, and form a regular collection, implying extremality in the upper core for $(2, 3, 3, 3, 12)$ and the other 19 allocations with the same coefficients.

Finally, The tight coalitions of the stable allocation $(0, 1, 7, 7, 8)$ form the regular collection $\{N, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$, implying extremality in the upper core of $(0, 1, 7, 7, 8)$ and the other 59 allocations with the same coefficients.

This shows that the mentioned allocations are the extreme upper core points. All allocations are feasible, implying that the game has a large core. Further, all collections of tight coalitions are non-degenerate, showing that the game is non-degenerate.

A game is called *strict exact* if for each coalition S a *core* allocation exist for which S and N are the only tight coalitions. Strict exactness implies strict upper exactness. To see this, let v be strict exact, and let S be an arbitrary coalition. A core allocation x exists for which S and N are the only tight coalitions. Then the sum of x and the indicator function of the complement of S, $x + e_{N \setminus S}$, is a stable allocation with S as the only tight coalition (this argument captures also the case $S = N$).

One easily derives the strict exactness of the game v'' in the previous example. This is not coincidental as the following result shows.

Proposition 8 *If a game is non-degenerate and strict upper exact, and has a large core, then it is strict exact.*

Proof: Let v be non-degenerate, strict upper exact, and let its core be large. For an arbitrary coalition S take a stable allocation x for which S is the only tight coalition. There is a core allocation x'' such that $x' \leq x$. Obviously, S and N are tight for x'. Since v is non-degenerate the collection of tight coalitions of x' has to be non-degenerate, and we may therefore assume that an $y \in I\!R^n$ exists such that $y(S) = y(N) = 0$ and $y(T) > 0$ for the other tight coalitions T of x'. For $\epsilon > 0$ sufficiently small the allocation $x'' = x' + \epsilon y$ belongs to the core of v. Its tight coalitions are S and N, thus showing that v is strict exact. □

We cannot leave out the non-degenerate property or the large core condition. This can be derived from the following two symmetric games w' and w'' on the player set $N = \{1, 2, 3\}$: $w'(S) = w''(S) = 0$ for coalitions S with 1 player, $w'(S) = w''(S) = 4$ if S consists of 2 players, and $w'(N) = 7$, and $w''(N) = 8$. It is left to the reader to check that both games are strict upper exact but not strict exact, w' is non-degenerate but does not have a large core, and w'' has a large core but fails to be non-degenerate.

Combining the previous two propositions we conclude that

Corollary 9 *A game is strict exact if its core consists of the maximum of $n!$ extreme points.*

5 Concluding remarks

Summarizing the contents of the paper, we proved that polyhedral sets of the form $P_{A,b} = \{x \in I\!R^n |\, Ax \geq b\}$ have at most $n!$ times the volume of the convex hull of the zero vector and the rows of the matrix A. We applied this result on 0,1-valued matrices and obtained the upper bound of $n!$ for the number of extreme points of the upper core and the core of a game. The maximum number is attained by the strict convex games but other games may have this property as well. These games have to be strict upper exact, must have a large core and fulfill a kind of non-degeneracy. We showed that not all games with these properties have $n!$ different extreme points. See also figure 1.

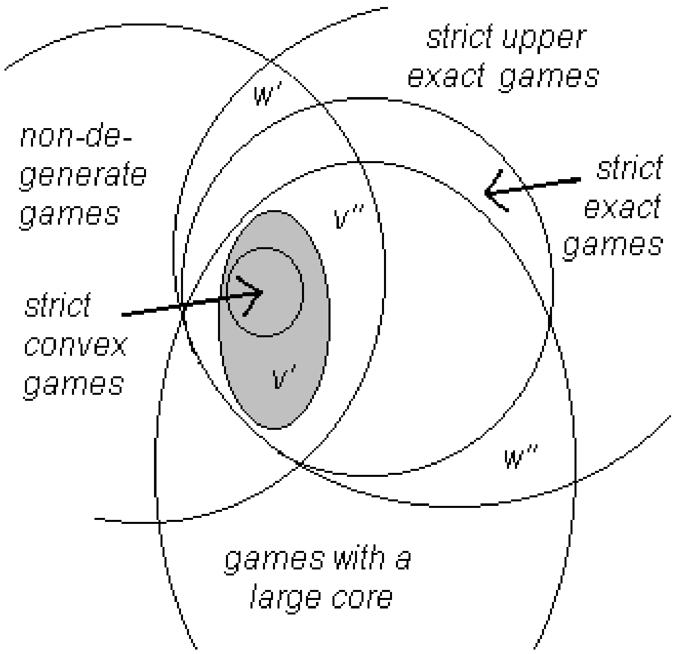

Figure 1: The grey area denotes the relative position of the games with maximum number of extreme core points.

Future research is concentrated on the dependence relations of the mentioned properties and on the impact of the non-degenerate condition which seems to involve combinatorial techniques for obtaining and analyzing the triangulations of the unit hypercube.

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