# On Cooperative Games, Inseparable by Semivalues *

Rafael Amer

*Department of Applied Mathematics II and Industrial Engineering School of Terrassa*
*Polytechnic University of Catalonia, Colom 11, E-08222 Terrassa, Spain.*
E-mail: amer@ma2.upc.es

Jean Derks

*Department of Mathematics, Universiteit Maastricht, Maastricht, The Netherlands*
E-mail: Jean.Derks@math.unimaas.nl

and

José Miguel Giménez

*Department of Applied Mathematics III and Polytechnic School of Manresa*
*Polytechnic University of Catalonia, Bases de Manresa 61, E-08240 Manresa, Spain.*
E-mail: jose.miguel.gimenez@upc.es

Semivalues like the Shapley value and the Banzhaf value may assign the same payoff vector to different games. It is even possible that two games attain the same outcome for all semivalues. Due to the linearity of the semivalues, this exactly occurs in case the difference of the two games is an element of the kernel of each semivalue. The intersection of these kernels is called the shared kernel, and its game theoretic importance is that two games can be evaluated differently by semivalues if and only if their difference is not a shared kernel element.

The shared kernel is a linear subspace of games. The corresponding linear equality system is provided so that one is able to check membership. The shared kernel is spanned by specific $\{-1, 0, 1\}$-valued games, referred to as shuffle games. We provide a basis with shuffle games, based on an a-priori given ordering of the players.

*Key Words:* Cooperative game, semivalue, shared kernel, shuffle game

## 1. INTRODUCTION

Semivalues were introduced by Dubey, Neyman and Weber in 1981 and reconsidered by Weber in 1988, and Carreras and Freixas in 1999 among many other authors. They represent a natural generalization of both the Shapley value (Shapley, 1953) and the Banzhaf value (Owen, 1975). The semivalues form a subset of the so called probabilistic values. These values evolve from a probabilistic pay off process where each of the players possesses a probabilistic distribution over the coalitions he is member of. If according to this distribution a coalition is chosen then the payoff to the player is the marginal contribution of that player to the chosen coalition. The corresponding probabilistic value is obtained as the expected payoff of the players.

Interesting probabilistic values are the weighted semivalues, where each of the probability weights is a product of a player dependent weight and a coalition dependent weight (Calvo and Santos, 1999), and the semivalues, where the probability weights are only dependent on the coalition size.

The probabilistic values form a wide family of solutions. One indication of its largeness is the dimension of the corresponding parameter space, and its ability to reveal differences among games. It is easily shown, for example, that for any two different games there is a weighted semivalue with different payoff vectors for these games.

The family of semivalues, however, does not have this property. It may happen that different games attain the same payoff vector, whatever semivalue is chosen. This aspect of inseparability is examined here. Due to the linearity property of the semivalues inseparability between two different games exactly occurs in case the difference of the two games is an element of the intersection of the kernels of all semivalues. This is a linear subspace of games which we call the shared kernel. We show that its dimension is $2^n - n^2 + n - 2$, with $n$ denoting the number of players. The corresponding linear equality system is provided so that one is able to check membership. The shared kernel is spanned by specific $\{-1, 0, 1\}$-valued games, referred to as shuffle games. We provide a basis with shuffle games, based on an a-priori given ordering of the players.

## 2. PRELIMINARIES

A cooperative *game* with transferable utility or TU-game is a pair $(N, v)$, or $v$ for short, where $N$ is a finite set of *players* and $v : 2^N \longrightarrow \mathbb{R}$ is the so called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *gain* or *worth* of coalition $S$, and satisfies the natural condition $v(\emptyset) = 0$.

With $G_N$ we denote the set of all cooperative games on $N$. It is an $2^n - 1$-dimensional Euclidean vector space, with $n$ the cardinality, $|N|$, of the player set.

For a non empty coalition $S \subseteq N$, the *unity game* $1_S$ is defined as follows:

$$1_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

The $2^n - 1$ unity games form a basis of the vector space $G_N$.

The *payoff vector* space $\mathbb{R}^N$ is also called the *allocation space*. A function $\psi : G_N \longrightarrow \mathbb{R}^N$ is called a *solution*; it represents a method to measure the negotiation strength of the players in the game. It can also be interpreted as an evaluation method for participation in the game.

Well studied solutions are the probabilistic values. Let $p = ((p^i_T)_{T\subseteq N, T\ni i})_{i\in N}$ be a collection of probability distributions, for each player one (so, the weights are non-negative and $\sum_{T\subseteq N, T\ni i} p^i_T = 1$ for each $i \in N$). Consider the following probabilistic payoff procedure, say in the game $(N, v)$: each player $i$, after choosing a coalition $S$ in accordance with his probability distribution $p^i$, is paid his marginal contribution $v(S) - v(S\backslash\{i\})$. Then the expected payoff to the players is called the *probabilistic value* $\psi^p$:

$$\psi^p_i[v] = \sum_{S\subseteq N, i\in S} p^i_S[v(S) - v(S\backslash\{i\})], \qquad i \in N.$$

Well known solutions like the Shapley value and the Bhanzaf value are probabilistic values. In the case of the Bhanzaf value each coalition has an equal probability to be chosen, namely $1/(2^{n-1})$. In the case of the Shapley value coalitions of the same size have equal probability, and each size is chosen with the same probability, yielding probability $1/(n\binom{n-1}{s-1})$ for a coalition of size $s$.

A probabilistic value corresponding to a weight system where coalitions of the same size have equal probability is called a *semivalue*. This family of solutions is obviously parameterized by the weight systems $p = (p_s)^n_{s=1}$ for which

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1, \quad \text{and } p_s \geq 0, 1 \leq s \leq n. \tag{1}$$

hold. Due to the fact that in a semivalue the equal sized coalitions are treated equally one might suggest that the mean worths of equal sized coalitions play a dominant role in the determination of the semivalue of a game. This is indeed the case as we will show now. For mathematical convenience we will use the notion of total accumulated worth in stead of the similar notion of mean worth.

For any cooperative game $v \in G_N$ we define the following amounts:

$$a_{i,s}(v) = \sum_{S\ni i, |S|=s} v(S) \qquad \text{for all } i \in N \text{ and } 1 \leq s \leq n.$$

$a_{i,s}$ denotes the total worth, accumulated over all coalitions of size $s$, containing player $i$ (see Dragan, 1992, for an application in the context of the Shapley value).

PROPOSITION 2.1. *For the computation of the semivalues of a game $v$ only the worths $a_{i,s}(v)$, $i \in N$ and $1 \leq s \leq n$, are needed.*

*Proof.* For the semivalue $\psi^p$ with coeficients $p_1, p_2, \ldots, p_n$, we have

$$\begin{aligned}
\psi_i[v] &= \sum_{S \ni i} p_s \left[v(S) - v(S\backslash\{i\})\right] \\
&= p_1 v(\{i\}) + \sum_{s=2}^{n} p_s \left[ \sum_{S\ni i, |S|=s} v(S) - \sum_{S \ni i, |S|=s} v(S\backslash\{i\}) \right] \\
&= p_1 a_{i,1}(v) + \sum_{s=2}^{n} p_s \left[ a_{i,s}(v) - \sum_{S \not\ni i, |S|=s-1} v(S) \right] \\
&= p_1 a_{i,1}(v) + \sum_{s=2}^{n} p_s \left[ a_{i,s}(v) - \sum_{|S|=s-1} v(S) + \sum_{S\ni i, |S|=s-1} v(S) \right] \\
&= p_1 a_{i,1}(v) + \sum_{s=2}^{n} p_s \left[ a_{i,s}(v) + a_{i,s-1}(v) - \frac{1}{s-1} \sum_{j=1}^{n} a_{j,s-1}(v) \right] . (2)
\end{aligned}$$

This proves the Proposition. ▮

## 3. THE SHARED KERNEL

We say that two games $v, v' \in G_N$ are *inseparable by semivalues*, and write $v \sim v'$, if and only if $\psi[v] = \psi[v']$ for every semivalue $\psi$ on $G_N$. The game space is therefore divided into equivalence classes, i.e., maximal sets of games for which all semivalues attain the same outcome.

Semivalues are linear functions on the game space, so that $v \sim v'$ if, and only if, the game difference, $v - v'$, attains the zero payoff vector in all semivalues. Consequently, inseparability is directly related to the class of games $v \in G_N$ such that $\psi[v] = 0$ for every semivalue $\psi$ defined on $G_N$. We denote this set with $C_N$ and call it the *shared kernel* since it equals the intersection of the kernels of all semivalues. Because of the linearity of the semivalues $C_N$ is a linear subspace of $G_N$.

Given a game $v \in G_N$ the set of all the semivalues of this game is a convex subset of $\mathbb{R}^N$, as can be deduced easily from (1) and (2). Obviously, this allocation set coincides with the corresponding set of any game in the equivalence class of $v$. The converse, however, is not true as the following example shows.

Let $N = \{1, 2, 3\}$ and $v, w \in G_N$ be given by $v = 1_{\{1\}} + 1_{\{2\}} + 1_{\{3\}}$, and $w = 1_N$. One easily derives from (2) that both games have the same set of semivalues: the convex hull of the zero payoff vector and $(1, 1, 1)$. $v$ and $w$ do not belong to the same equivalence class, however, since for example the Shapley value of $v - w$ does not equal the zero payoff vector.

The equations in (2) provide us a method to determine whether a game is an element of the shared kernel, and thereby a method of determining inseparability between games.

PROPOSITION 3.1. *The game $v \in G_N$ belongs to the shared kernel $C_N$ if, and only if,*

$$a_{i,s}(v) = 0 \qquad \textit{for all } i \in N \textit{ and } 1 \leq s \leq n.$$

*Proof.* The 'If'-part follows directly from the fact that if $a_{i,s}(v) = 0$ for all $i \in N$ and $1 \leq s \leq n$, then (2) implies $\psi_i[v] = 0$ for all $i \in N$.

To prove the 'Only if'-part assume that $\psi^p[v] = 0$ for all semivalues $\psi^p$. Let $\psi^s$ denote the semivalue with weight system

$$p_s = \frac{1}{\binom{n-1}{s-1}}\,, \qquad p_t = 0 \quad \text{otherwise.}$$

Then, for $i \in N$, we have $0 = \psi_i^1[v] = a_{i,1}(v)$ and, for any $s$ such that $2 \leq s \leq n$,

$$a_{i,s}(v) + a_{i,s-1}(v) - \frac{1}{s-1}\sum_{j=1}^{n} a_{j,s-1}(v) = 0$$

since $\psi_i^s[v] = 0$. It is easy to see that these equalities imply that $a_{i,2}(v) = \cdots = a_{i,n}(v) = 0$ for all $i \in N$. ∎

COROLLARY 3.1. *Two games $v, v' \in G_N$ are inseparable by semivalues if, and only if, $a_{i,s}(v) = a_{i,s}(v')$ for all $i \in N$ and $1 \leq s \leq n$.*

THEOREM 3.1. *The dimension of the shared kernel is $2^n - n^2 + n - 2$.*

*Proof.* The $n$ restrictions $a_{i,n}(v) = 0$, for $1 \leq i \leq n$, are equivalent to the single condition $v(N) = 0$; thus, a total of $n(n-1)+1$ restrictions appear that affect the components of the vectors of the space of games $G_N$, whose dimension is $2^n - 1$. Then, it will be enough to prove that these restrictions are linearly independent equations to affirm that

$$\dim C_N = 2^n - 1 - [n(n-1)+1] = 2^n - n^2 + n - 2\,.$$

In order to verify the linear independence of the restrictions, we put together these restrictions according to the cardinality of the coalitions to which they affect:

$$a_{i,1}(v) = 0 \text{ for } 1 \leq i \leq n\,; \ldots ; a_{i,n-1}(v) = 0 \text{ for } 1 \leq i \leq n\,; \quad v(N) = 0.$$

As is obvious, any restriction concerning a specific cardinality is independent of all other restrictions affecting to different cardinalities.

The restrictions corresponding to a given cardinality $s$ $(1 \leq s \leq n-1)$ form a system of $n$ linear equations with $\binom{n}{s}$ unknowns. We need to determine the rank of the matrix $B = (b_{i,S})$ where $i$ ranges over all players, and $S$ ranges over all coalitions of cardinality $s$, with $b_{i,S} = 1$ if $i \in S$ and 0 otherwise.

We choose the columns corresponding to the following coalitions: the consecutive coalitions with first player $1, 2, \ldots, n-s-1$, all coalitions with first player $n-s$ and the consecutive coalition with first player $n-s+1$. The determinant of this submatrix is $(-1)^{\mathrm{int}(s/2)} s$, thus the rank of $B$ is $n$ and this guarantees the linear independence of the restrictions that affect to each one of the coalition cardinalities, for $1 \leq s \leq n-1$. ▮

For each coalition cardinality, from 1 to $n-1$, $n$ independent linear restrictions appear, so that the freedom degrees turn out to be

| **Cardinality** | $s=1$ | $s=2$ | $\cdots$ | $s=n-2$ | $s=n-1$ | $s=n$ |
|---|---|---|---|---|---|---|
| freedom degrees | $\binom{n}{1}-n=0$ | $\binom{n}{2}-n$ | $\cdots$ | $\binom{n}{n-2}-n$ | $\binom{n}{n-1}-n=0$ | $1-1=0$ |

For cardinalities $1$, $n-1$ and $n$, the number of independent restrictions agrees with the number of unknowns, so that the only solution is the null solution. The sum of all freedom degrees will give the dimension of the shared kernel:

$$\dim \mathcal{C}_N = \sum_{s=2}^{n-2} \left[ \binom{n}{s} - n \right] = \sum_{s=2}^{n-2} \binom{n}{s} - (n-3)n = 2^n - n^2 + n - 2\,.$$

For small values of $n$, we have

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\dim C_N$ | 0 | 0 | 2 | 10 | 32 | 84 | $\cdots$ |
| $\dim G_N$ | 3 | 7 | 15 | 31 | 63 | 127 | $\cdots$ |

Then, for $n = 2$ and $n = 3$ all cooperative games are separable by semivalues whereas, for cardinality $n \geq 4$, the shared kernel is a linear subspace of $G_N$ and its supplementaries have dimension $n^2 - n + 1$.

Let us call the number of coalitions with non-zero worth in a game $v$ the *caliber* of $v$. The caliber of a game in the shared kernel is either 0 (the all zero game) or at least 4. To show this, let $v \in C_N$ with caliber unequal to 0, say coalition $S$ has worth unequal to zero. Let $i \in S$ be a member of $S$, and $s = |S|$. From $a_{i,s}(v) = 0$ we derive that there is another coalition, say $S'$, with size $s$ and containing $i$ and with $v(S') \neq 0$. Take a player $j \in S' \backslash S$. From $a_{j,s}(v) = 0$ we again derive the existence of a coalition, say $S''$, with size $s$, containing $j$ and with $v(S'') \neq 0$. Now suppose these three different coalitions are the only ones with nonzero worth in $v$. From $0 = \sum_{k \in N} a_{i,s} = s(v(S) + v(S') + v(S''))$ and $0 = a_{j,s}(v) = v(S') + v(S'')$ we must conclude that $v(S) = 0$; a contradiction.

## 4. A BASIS FOR THE SHARED KERNEL

In the following we consider the cardinality of the player set $N$ to be at least 4, $n \geq 4$. For a given coalition $S \subseteq N$ and players $i, j \in S$ and $k, l \in N \backslash S$, we define the *shuffle game* $v_{S,i,j,k,l}$ as

$$v_{S,i,j,k,l} = 1_S + 1_{S \cup \{k,l\} \backslash \{i,j\}} - 1_{S \cup \{k\} \backslash \{i\}} - 1_{S \cup \{l\} \backslash \{j\}} \,.$$

If $v \in G_N$ is a shuffle game, it is evident that $a_{i,s}(v) = 0$ for $1 \leq i \leq n$ and $1 \leq s \leq n$. According to Proposition 3.1, we may conclude that $v$ is an element of the shared kernel, i.e., the shuffle games are contained in the shared kernel. Loosely speaking, there are no other games:

THEOREM 4.1. *The shared kernel is spanned by shuffle games.*

To ease the mathematical argumentation in the proof of the Theorem we assume, without loss of generalization, that the players are ordered: $N = \{1, 2, \ldots, n\}$. We can therefore speak of a last and a first player in a coalition $S$, denoted by $l(S)$ and $f(S)$. Further, we say that a player is absent in a coalition if he is positioned between the first and last player of that coalition but is not a member. Consecutive coalitions denote coalitions without absent players.

For coalitions $S$ with $l(S) \geq |S| + 2$, we call the last player, not in $S$, but positioned right before $l(S)$, $f'(S)$, and the second last player $l'(S)$. For these coalitions $S$, i. e., $|S| \geq 2$ and $l(S) \geq |S| + 2$, we consider the shuffle game

$$\begin{aligned} v_S = v_{S,f(S),l(S),f'(S),l'(S)} = & 1_S + 1_{S \cup \{f'(S),l'(S)\} \backslash \{f(S),l(S)\}} \\ & - 1_{S \cup \{f'(S)\} \backslash \{f(S)\}} - 1_{S \cup \{l'(S)\} \backslash \{l(S)\}} \,. \end{aligned}$$

The choice of this shuffle game is such that we can write a unity game $1_S$, where coalition $S$ has 2 or more absent players, as a weighted sum consisting of the shuffle game $v_S$ and three unity games corresponding to coalitions with less absent players than in $S$. In particular, for a coalition $S$ with only one absent player, and $l(S) \geq |S| + 2$, we see that $1_S$ is a weighted sum consisting of the shuffle game $v_S$, the unity game corresponding to the consecutive coalition with cardinality $|S|$ and first player $f(S) + 1$ and two unity games corresponding to coalitions with only one absent player but where last player is $l(S) - 1$.

Since a game is a weighted sum of unity games we conclude from the previous arguments that it can be written as a weighted sum of shuffle games and unity games, both corresponding to consecutive coalitions or coalitions $S$ having only one absent player and $l(S) = |S| + 1$. In other words, any game $v$ can be written as the sum of a game from the span of the shuffle games and a rest-game $v^*$ with the property that $v^*(S) \neq 0$ implies that $S$ has no absent player, or only one absent player and last player $|S| + 1$.

*Proof.* (*Theorem 4.1*) Now, let $v$ be a member of the shared kernel. The resulting rest-game $v^*$ should also be a member, due to the fact that the shared kernel is a linear subspace. Therefore

$$a_{i,s}(v^*) = \sum_{S \ni i, |S| = s} v^*(S) = 0 \qquad \text{for } 1 \leq i \leq n \text{ and } 1 \leq s \leq n. \tag{3}$$

We will prove that this implies that $v^*$ can only be the all-zero game.

We distinguish three cases (where the first two applies to any game in the shared kernel).

- Applying the cases $s = 1$ and $s = n$, it follows immediately that $v^*$-values of the one-person coalitions and the grand coalition $N$ are zero.
- Consider now the case $s = n - 1$. We have

$$\sum_{j \neq i} v^*(N \backslash \{j\}) = 0 \qquad \text{for all } i \in N,$$

and this is only possible if $v^*(N \backslash \{j\}) = 0$ for all $j \in N$.

- Consider the remaining cardinalities $1 < s < n - 1$. A coalition $S$ with $|S| = s$ and non-zero values in $v^*$ is either consecutive or the coalition has one absent player and $l(S) = s + 1 < n$. Therefore the only possible coalitions of size $s$, containing one of the players $s + 2, \ldots, n$, and non-zero valued in $v^*$, are the consecutive ones. There is only one such coalition containing $n$, so that by (3) this coalition is zero valued in $v^*$. One easily shows by induction that also other consecutive coalitions containing one of the players $s + 1, \ldots, n$ are zero valued.

The only coalitions left to examine are the ones with cardinality $s$ and last player smaller than or equal to $s + 1$. Similarly to the case $s = n - 1$ one may conclude that these coalitions are also zero valued in $v^*$.

So we end up with an all-zero game $v^*$, implying that the shared kernel is spanned by the shuffle games. ∎

Observe that we only used the games $v_S = v_{S,f(S),l(S),f'(S),l'(S)}$ as defined above with respect to a given ordering. We actually used the ones corresponding to coalitions $S$ with cardinality $s$ between 2 and $n - 2$, and having at least two absent players, or with one absent player and last player equal to $s + 2$ or larger.

The total number of used shuffle games are $2^n - 1$ minus the number of coalitions that do not match the above description. These are:
• all coalitions of size 1, $n$-1, and $n$ (total of $2n + 1$);
• all consecutive coalitions of size between 2 and $n - 2$ (total of $(n - 1) + (n - 2) + \ldots + 3$);
• all coalitions of size $s$ between 2 and $n - 2$, with one absent player and last player equal to $s + 1$ (total of $1 + 2 + \ldots + (n - 3)$).
Adding these numbers we arrive at a total of $2n + 1 + (n - 3)n = n^2 - n + 1$, concluding that we have used $2^n - 1 - (n^2 - n + 1)$ shuffle games $v_S$ or less in the above derivation of the rest game $v^*$. This (maximal) number equals the dimension of the shared kernel so that we may conclude that

THEOREM 4.2. *The shuffle games $v_S$, as defined above, for coalitions $S$ with size $s$ between 2 and $n - 2$, having at least two absent players, or with one absent player and last player equal to $s + 2$ or larger, form a basis of the shared kernel.*

## REFERENCES

1. Calvo, E., and Santos, J.C., “Weighted weak semivalues”, *International Journal of Game Theory* **29**, (2000) p. 1–9.
2. Carreras, F., and Freixas, J., “Some theoretical reasons for using (regular) semivalues”, *Logic, Game Theory and Social Choice* (H. de Swart, Tilburg University Press, 1999), p. 140–154.
3. Dragan, I., “An average per capita formula for the Shapley value”, *Libertas Mathematica* **12**, (1992) p. 139–146.
4. Dubey, P., Neyman, A., and Weber, R. J. “Value theory without efficiency”, *Mathematics of Operations Research*, **6**, (1981) p. 122–128.
5. Owen, G., “Multilinear extensions and the Banzhaf value”, *Naval Research Logistics Quarterly*, **22**, (1975) p. 741–750.
6. Shapley, L. S., “A value for n-person games”, *Contributions to the Theory of Games II* (H. W. Kuhn and A. W. Tucker, Princeton University Press, 19??), p. 307–317.
7. Weber, R. J., “Probabilistic values for games”, *The Shapley value: Essaiys in honor of Lloyd S. Shapley*, (A. E. Roth, Cambridge University Press, 1988) p. 101–119.