A short proof of the inclusion of the Core in the Weber set

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Abstract

For a transferable utility game the convex hull of the Weber allocations contains the core as a subset. A proof of this result normally involves an induction hypothesis to the number of players. We will give here a short proof based on a well known result in convex analysis.

Running title: The inclusion of the Core in the Weber set

For a transferable utility game v on the player set $N = \{1, 2, \ldots, n\}$ consider the so called Weber allocation $x^\pi \in I\!R^N$, with π a permutation of N, defined by

$$x^\pi_{\pi(i)} = v(\{\pi(1), \pi(2), \ldots, \pi(i)\}) - v(\{\pi(1), \pi(2), \ldots, \pi(i-1)\}).$$

In Weber(1978) it is shown that the convex hull of these allocations, denoted by

$$W(v) = \text{Conv}(\{x^\pi \colon \pi \text{ permutation of } N\}),$$

contains each core allocation of v, i.e., $x \in W(v)$ if $\sum_{j \in S} x_i \geq v(S)$ for all $S \subseteq N$, and $\sum_{i \in N} x_i = v(N)$. A proof of this result normally involves an induction hypothesis on the number of players in N. We will give here a

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short proof based on a well known result in convex analysis.

Theorem The Core is a subset of the Weber set.

Proof: Let us suppose that there is a core allocation, say x, not contained in the Weber set $W(v)$. According to the Separation Theorem, the convexity and closedness of $W(v)$ implies that there exists a vector $y \in I\!R^N$ such that the inproduct $z \cdot y$ strictly exceeds $x \cdot y$ for each $z \in W(v)$. In particular,

$$x^\pi \cdot y > x \cdot y \quad \text{for each permutation } \pi. \tag{1}$$

Let the permutation π of N be such that $y_{\pi(1)} \geq y_{\pi(2)} \geq \ldots \geq y_{\pi(n)}$. Then

$$\begin{aligned}
x^\pi \cdot y &= \sum_{i=1}^{n} y_{\pi(i)}(v(\{\pi(1), \pi(2), \ldots, \pi(i)\}) - v(\{\pi(1), \pi(2), \ldots, \pi(i-1)\})) \\
&= y_{\pi(n)} v(N) + y_{\pi(1)} v(\emptyset) + \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) v(\{\pi(1), \pi(2), \ldots, \pi(i)\}) \\
&\leq y_{\pi(n)} \sum_{j=1}^{n} x_{\pi(j)} + \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) \sum_{j=1}^{i} x_{\pi(j)} \\
&= \sum_{i=1}^{n} y_{\pi(i)} \sum_{j=1}^{i} x_{\pi(j)} - \sum_{i=2}^{n} y_{\pi(i)} \sum_{j=1}^{i-1} x_{\pi(j)} \\
&= \sum_{i=1}^{n} y_{\pi(i)} x_{\pi(i)}
\end{aligned}$$

and this is contradictory to (1). We conclude that a core allocation has to be an element of the Weber set. □

In Derks, Gilles (1992) similar arguments are used in the proof of a generalization of the above theorem for games defined on a collection of coalitions with a lattice structure.

References

[1] Derks JJM, Gilles RP (1992) *Hierarchical Organization Structures and Constraints in Coalition Formation.* Working Paper, Department of Economics, Virginia State University Blacksburg

[2] R.J. Weber (1978) *Probabilistic values for games* Cowles Foundation Discussion Paper 417R, Yale University, New Haven.